J. K. Wakefield
20 Felton.

SOLID GEOMETRY

AND

CONIC SECTIONS.

"Ὡς οἷόν τ' ἄρα, ἦν δ' ἐγώ, μάλιστα προστακτέον, ὅπως οἱ ἐν τῇ καλλιπόλει σοι μηδενὶ τρόπῳ γεωμετρίας ἀφέξονται· πρὸς γὰρ πάσας μαθήσεις, ὥστε κάλλιον ἀποδέχεσθαι, ἴσμεν που ὅτι τῷ ὅλῳ καὶ παντὶ διοίσει ἡμμένος τε γεωμετρίας καὶ μή. τῷ παντὶ μέντοι νὴ Δί', ἔφη."

PLATO, *Repub.* Bk. VII. 527.

This was Diuine Plato his Iudgement, both of the purposed, chief, and perfect vse of Geometrie; and of his second, dependyng and deriuatiue commodities. And for vs, Christen men, a thousand thousand mo occasions are to haue nede of the helpe of Megethologicall Contemplations; wherby to trayne our Imaginations and Myndes, by litle and litle, to forsake and abandon the grosse and corruptible Objectes of our vtward senses: and to apprehend, by sure Doctrine demonstratiue, Things Mathematicall.

John Dee his Mathematicall Preface to Euclides Elementes.
A. D. 1570.

SOLID GEOMETRY

AND

CONIC SECTIONS,

WITH APPENDICES ON TRANSVERSALS, AND HARMONIC DIVISION,

FOR THE USE OF SCHOOLS,

BY

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PREFACE.

THIS work is an endeavour to introduce into schools some portions of Solid Geometry which are now very little read in England. The first twenty-one Propositions of Euclid's Eleventh Book are usually all the Solid Geometry that a boy reads till he meets with the subject again in the course of his analytical studies. And this is a matter of regret, because this part of Geometry is specially valuable and attractive. In it the attention of the student is strongly called to the subject matter of the reasoning; the geometrical imagination is exercised; the methods employed in it are more ingenious than those in Plane Geometry, and have greater difficulties to meet; and the applications of it in practice are more varied.

I have added short Appendices on Transversals, and on Harmonic Division, which will, I hope, be found useful.

In the chapters on Conic Sections I have endeavoured to shorten the subject, which, as presented in the most extensively used text-books, those of Drew, Taylor, and Besant, has somewhat outgrown the capabilities of schoolboys. This is accomplished partly by defining these curves as sections of a cone, and deducing immediately their fundamental properties; and partly also by taking

the ellipse and hyperbola together in proving many of their common properties. I have tried in all cases to give the most natural proof, and to arrange the Theorems in an easily remembered order, and have used Corollaries freely. I have also left out the subject of Curvature, which may be read by the student along with Newton with greater advantage than at this early stage of his studies.

RUGBY, *January*, 1872.

PREFACE TO THE SECOND EDITION.

IN this edition there has been some change in Section I, and Section IV. has been largely added to. There have been also some slight changes in the chapters on the ellipse and hyperbola.

I have much pleasure in acknowledging my obligations to Mr R. Tucker for corrections and criticism.

RUGBY, *January*, 1873.

PREFACE TO THE THIRD EDITION.

IN this edition I have added an important theorem on the parabola, and a few exercises, but have made no other change.

RUGBY, *Feb.* 1876.

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SECTION I.

PLANES.

STRAIGHT LINES AND POINTS IN A PLANE.

Def. 1. A *Plane* is a surface in which any two points being taken, the straight line which joins them lies wholly in that surface.

Def. 2. A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line in that plane. It is also said to be *normal* to the plane.

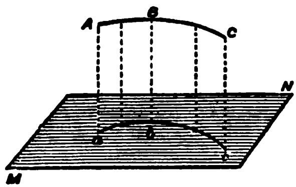
Def. 3. A straight line is said to be *parallel to a plane*, when if indefinitely produced both ways it does not meet the plane.

Def. 4. Planes which never intersect one another, however far they are produced, are called *parallel* planes.

They are then said to have the same *disposition* in space.

Def. 5. The *projection of a point on a plane* is the foot of the perpendicular let fall from the point to the plane.

Def. 6. The *projection of a line on a plane* is the locus of the feet of the perpendiculars let fall from the points in that line to the plane.



Thus a is the projection of A on the plane MN, and the line abc is the projection of ABC on the same plane[1].

Def. 7. The angle which a straight line makes with its projection on any plane is called the *angle which the straight line makes with the plane.*

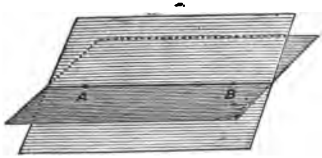
Def. 8. The inclination of two straight lines in space which do not intersect is the angle formed by straight lines from any point respectively parallel to those lines.

Def. 9. The angle between two planes is called a *dihedral angle.*

A dihedral angle is measured by the angle between the two lines, drawn one in each plane perpendicular to the line of intersection of the planes from any point in that line.

THEOREM I.

The line of intersection of two planes is a straight line.



[1] Projections were first used in geometrical investigations by Desargues in 1640, and their use very greatly extended by Monge and Poncelet.

Proof. Let A, B be two points in the line of intersection of two planes; join A, B.

Then, by Def. 1, the straight line AB lies in both planes; therefore the straight line AB is the line of intersection of the two planes[1].

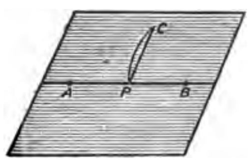
THEOREM 2.

Through two given points an indefinite number of planes may be drawn.

Proof. A plane may be conceived as revolving round the straight line joining the two given points, and as occupying in succession an indefinite number of positions.

THEOREM 3.

Through three given points, not in the same straight line, one plane and only one plane may be drawn.



Proof. Let A, B, C be the three given points.

Conceive a plane to revolve round AB; then in some position it will pass through C; and therefore a plane may be drawn to contain A, B, and C.

Further, only one such plane can be drawn.

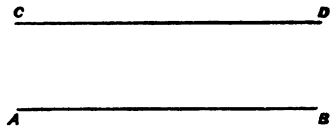
For if there were two, then there could be drawn from C two straight lines CP, one in each plane, to any point P in the line AB, which is absurd.

COR. *Through two intersecting straight lines one plane and only one plane can be drawn.*

[1] Euclid XI. 3.

THEOREM 4.

One plane and only one plane can be drawn to contain two parallel straight lines.



Proof. Let AB, CD be the parallels.

Conceive a plane to revolve round AB till it passes through C.

Then since parallel lines are in the same plane, the line CD will be in the plane that contains AB and C.

Note. From these Theorems we learn that a plane is determined,

(1) by three points not collinear;

(2) by two intersecting straight lines;

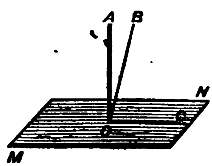
(3) by two parallel lines.

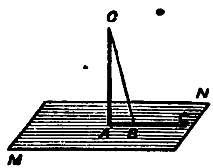
THE STRAIGHT LINE PERPENDICULAR TO THE PLANE.

THEOREM 5.

There cannot be two perpendiculars drawn to a given plane from a point either in the plane or without it.

Proof. If possible let OA, OB be both perpendicular to the plane MN.





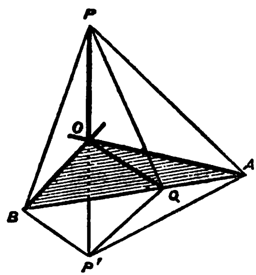
Since the plane determined by OA, OB intersects the plane MN in a straight line (Th. 1);

then OA, OB would each of them be perpendicular to that line (by Def. 2) and in the same plane with it; which is impossible.

THEOREM 6.

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it will be perpendicular to the plane which contains them.

Proof. Let OP be perpendicular to each of OA, OB, at their point of intersection O.



Draw any line from O, in the plane containing OA, OB, meeting AB in Q; then will PO be perpendicular to OQ.

Produce PO to P', making $OP' = OP$.

Join P, P', with A, Q, and B.

Since AO is perpendicular to PP' through its middle point O,

therefore $AP = AP'$:

similarly $BP = BP'$:

therefore the triangles APB, $AP'B$ are equal:

and therefore if $AP'B$ were conceived to revolve round AB, till the planes of the triangles coincided, P' would fall on P, and QP' on QP.

Therefore $QP' = QP$.

And therefore in the triangles QOP, QOP', since the three sides of the one are respectively equal to the three sides of the other, the angle $QOP =$ the angle QOP';
and therefore QOP is a right angle;
that is, PO is at right angles to every straight line that meets it in the plane AOB[1].

Therefore PO is perpendicular to the plane AOB.

COR. 1. *Of all the straight lines that can be drawn to a given plane from a given point, the shortest is the perpendicular; and of the others those whose extremities are equally distant from the foot of the perpendicular are equal, and conversely.*

For, firstly, since PP' is $< PB + BP'$,
therefore PO is $< PB$.

And again if $OA = OB$, then the triangles POA, POB have two sides and the included angles equal, and therefore $PB = PA$.

Conversely, if $PB = PA$ in the right-angled triangles POB, POA, then $OB = OA$.

COR. 2. *The locus of points equally distant from two given points is the plane that bisects at right angles the line joining the two points.*

COR. 3. *The locus of straight lines which cut a given straight line at right angles at a given point is a plane.*

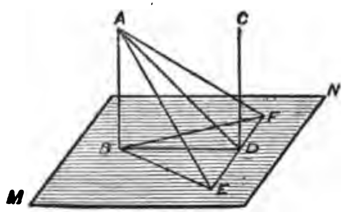
Note. Hence a plane is determined by one point, and the normal to the plane at that point.

[1] This proof is due to Legendre.

THEOREM 7.

If two straight lines are parallel, and one of them is perpendicular to a plane, the other will be also perpendicular to that plane, and conversely[1].

Let AB, CD be parallel straight lines, and let AB be perpendicular to the plane MN;



then will CD be also perpendicular to MN.

Join AD, BD; and in the plane MN draw EDF at right angles to BD, and take $DE = DF$.

Join BE, BF, AE, AF.

Then since BE, BF are obliques drawn from B to the line EF, equally distant from the perpendicular BD, therefore $BE = BF$. (Bk. I.)

Again, since AE, AF are obliques drawn from A to the plane MN equally distant from the perpendicular AB, therefore $AE = AF$. (Th. 6, Cor. I.)

But $ED = DF$; and AD is common to the two triangles AED, AFD, and therefore the angle ADE = the angle ADF; therefore ADE is a right angle;

that is, ED is perpendicular to DA.

Also ED is perpendicular to BD; (Constr.)

[1] Euclid XI. 8.

therefore ED is perpendicular to the plane ADB. (Th. 6.)

But since AB is parallel to DC, the plane ADB contains the line DC.

Therefore ED is perpendicular to DC, and CDE is a right angle. (Th. 6.)

Again, since AB is parallel to CD, and BD joins them, and ABD is a right angle; (Hyp.)
therefore CDB is also a right angle.

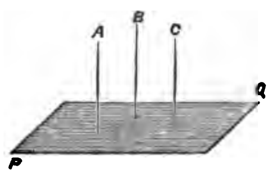
Hence CD is at right angles to both BD and DE; therefore CD is at right angles to the plane containing DB and DE; that is, to the plane MN. (Th. 6.)

Conversely, if AB and CD are both perpendicular to the plane MN, AB and CD will be parallel.

For there can be but one parallel to AB through D, and one perpendicular to the plane through D; and the parallel is the perpendicular by the first part of the Theorem: therefore the perpendicular is the parallel.

Theorem 8.

Straight lines which are parallel to the same straight line are parallel to one another[1].



Let A and B be each of them parallel to C; then shall A be parallel to B.

Proof. Take a plane PQ perpendicular to C.

Then since A is parallel to C, therefore A is perpendicular to PQ. (Th. 7.)

[1] Euclid XI. 9.

And again, since B is parallel to C,

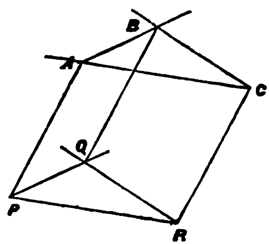
therefore B is perpendicular to PQ. (Th. 7.)

And since A and B are perpendicular to the same plane PQ,

therefore A is parallel to B. (Th. 7.)

THEOREM 9.

If two straight lines meeting one another are respectively parallel to two other straight lines meeting one another, they will include equal angles[1].



Let the lines AB, CB, meeting in B, be respectively parallel to the lines PQ, RQ meeting in Q; they will include equal angles.

From the lines BA, QP, in the same sense, cut off any equal parts BA, QP;

and from the lines BC, QR, in the same sense, cut off equal parts BC, QR.

Join AC, PR, AP, BQ, CR.

Proof. Because BA is equal and parallel to QP;
(Hyp. and Const.)

therefore BQ is equal and parallel to AP. (Bk. 1.)

And because BC is equal and parallel to QR; therefore BQ is equal and parallel to CR.

Therefore AP is equal and parallel to CR, (Th. 8.) and therefore also AC is equal and parallel to PR.

[1] Euclid XI. 10.

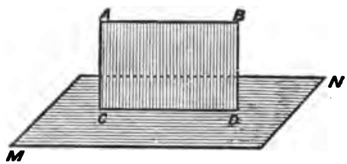
Therefore in the triangles ABC, PQR, the three sides of the one are respectively equal to the three sides of the other;

therefore the angle ABC = the angle PQR. (Bk. I.)

The Straight Line parallel to the Plane.

Theorem 10.

If a straight line is parallel to a line in a given plane, it is either parallel to that plane, or lies in that plane.



Proof. Let AB be parallel to CD, in the plane MN; then will AB be either parallel to MN, or lie in the plane MN.

Let a plane be drawn to contain the parallels AB, CD.

Then if this plane is identical with MN, AB lies in the plane MN; but if not, it cuts MN in the line CD.

And since AB is parallel to CD, AB will not meet CD, and therefore will not meet the plane MN, however far it is produced;

that is, AB is parallel to the plane MN.

Cor. *Through a given point an indefinite number of straight lines may be drawn parallel to a given plane.*

THEOREM 11.

If a straight line is parallel to a plane, it is also parallel to the line of intersection of any plane containing it with the given plane.

Proof. Let AB be parallel to the plane MN, and let the plane $ABCD$ contain AB; then AB will be parallel to CD, the intersection of MN and $ABCD$.

For the plane $ABCD$ cuts MN in CD: and CD is parallel to AB, since they are in the same plane $ABCD$, and do not meet one another.

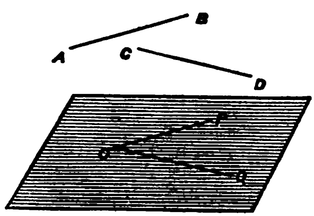
COR. 1. *If a straight line is parallel to a plane, the parallel to that straight line drawn through any point in the plane will lie in that plane.*

Proof. The parallel to AB through C is the line CD; and CD lies in the plane MN.

COR. 2. *If a straight line is parallel to two planes it is parallel to their line of intersection.*

Proof. For the parallel to the straight line through any point common to both planes lies in both planes, by Cor. 1, and therefore must be their line of intersection.

COR. 3. *Through a given point one plane and only one can be drawn parallel to two given lines.*



Proof. Let O be the given point, AB, CD the given lines. Draw OP, OQ parallel to AB, CD.

Then the plane determined by OP, OQ is parallel to both AB and CD by Theorem 10; and therefore one and only one plane through O is parallel to both AB and CD.

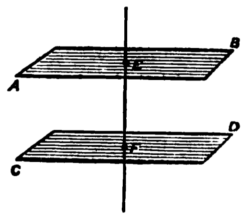
Note. Hence a plane is determined by one point, and the directions of two lines in the plane.

PARALLEL PLANES.

THEOREM 12.

Planes which have a common normal are parallel.

Proof. Let AB, CD have a common normal EF; then will AB and CD never meet.



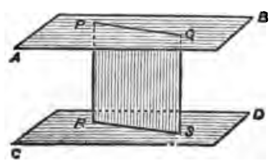
For if AB, CD had any point in common, from that point two perpendiculars could be drawn to EF, one in each plane, which is absurd.

Therefore AB and CD never meet: that is, they are parallel.

THEOREM 13.

If two parallel planes are intersected by a third plane, the lines of intersection will be parallel[1].

Proof. Let AB, CD be parallel planes, intersected by a third plane in PQ, RS; then PQ will be parallel to RS.



For PQ cannot meet RS, being in parallel planes;

[1] Euclid XI. 16.

and PQ and RS are in the same plane $PQRS$;
therefore PQ is parallel to RS.

COR. 1. *Hence two intersecting planes cannot both be parallel to the same plane.*

For they may be intersected by a third plane, so that of the three lines of intersection, two would intersect, and therefore could not both be parallel to the third.

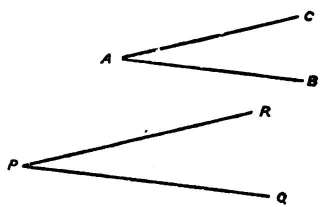
COR. 2. *Planes which are parallel have a common normal.*

COR. 3. *If parallel planes are intersected by parallel planes, the lines of intersection will be parallel.*

This may be otherwise stated, by saying, that by the dispositions of two planes the direction of their line of intersection is determined.

THEOREM 14.

If two intersecting lines in one plane are parallel respectively to two intersecting lines in another plane, the planes will be parallel[1].



Proof. Let AB, AC be respectively parallel to PQ, PR.

Then will the plane ABC be parallel to the plane PQR.

For by Th. 11, Cor. 3, the plane PQR is parallel to AB and AC, therefore by Th. 11, if PQR and ABC met, their

[1] Euclid XI. 15.

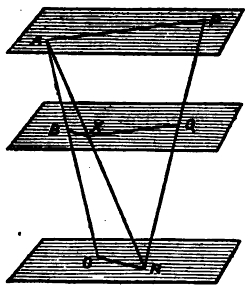
line of intersection would be parallel to both AB and AC, which is impossible.

Therefore the planes ABC, PQR are parallel.

COR. *Hence the directions of two lines in a plane determine the disposition of the plane.*

THEOREM 15.

If two straight lines are cut by three parallel planes, they will be cut proportionally.



Proof. Let three parallel planes be cut by ABC, PQR, then will the segments AB, BC, PQ, QR form a proportion.

Join AR, cutting the plane BQ in X, and join BX, XQ.

Then since the plane ACR intersects the parallel planes BQ, CR in BX, CR,

therefore BX is parallel to CR; (Th. 13.)

and therefore $AB : BC :: AX : XR$.

In the same manner since XQ is parallel to AP,

$$AX : XR :: PQ : QR;$$

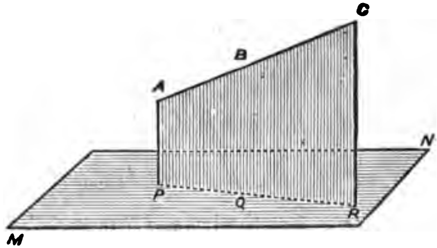
therefore $AB : BC :: PQ : QR$[1].

THE LINE INCLINED TO THE PLANE.

THEOREM 16.

The projection of a straight line on a plane is a straight line.

[1] Euclid XI. 17.



Proof. Let ABC be a straight line, MN a plane, and let P, Q, R be the feet of the perpendiculars from any three points A, B, C on the plane MN.

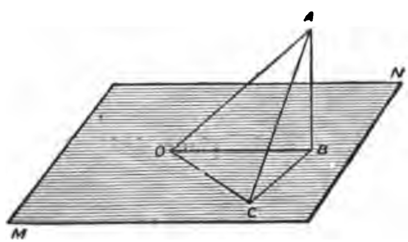
Then will PQR be a straight line.

For the plane containing AC and AP will contain all the perpendiculars (Th. 4), and it will intersect MN in a straight line PQR.

THEOREM 17.

The angle which a straight line makes with its projection on a plane is less than that which it makes with any other straight line that meets it in that plane. Except when [illegible] is $\parallel$ or $\perp$ to it.

Let OB be the projection of OA; OC any other line that meets it in the plane MN; then will AOB be $< AOC$.



Proof. Let AB be perpendicular to the plane MN, and take $OC = OB$, and join AC.

Then since AB is the perpendicular, and AC an oblique,

AB is $< AC$. (Th. 6, Cor. I.)

Therefore in the triangles AOB, AOC,

AO, $OB = AO$, OC; but the base AB is $< AC$,

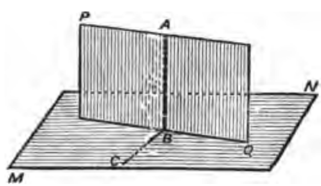
therefore the angle $AOB < AOC$,

which proves the proposition.

The Plane inclined to the Plane.

Theorem 18.

Every plane passing through a perpendicular to a plane is also perpendicular to that plane, and conversely[1].



Proof. Let AB be perpendicular to MN, and let the plane PQ contain AB; then will PQ be perpendicular to MN.

From B draw, in the plane MN, BC perpendicular to BQ.

[1] Euclid XI. 18.

Then since AB is perpendicular to the plane MN,

therefore AB is at right angles to both BQ and BC;

but the angle between the planes is measured by the angle ABC; (Def. 9.)

therefore the dihedral angle is a right angle.

Conversely, if PQ is at right angles to MN, PQ will contain the perpendicular to the plane drawn from any point B in the line of intersection.

For draw BA at right angles to QB in the plane PQ; then ABC measures the dihedral angle;

therefore ABC is a right angle;

that is, AB is at right angles to BQ and BC;

and therefore AB is perpendicular to the plane MN.

COR. *If two planes are perpendicular to a third plane, their line of intersection will be also perpendicular to that plane.*

For the perpendicular from the point of intersection of the planes must lie in both the planes, and therefore be their line of intersection.

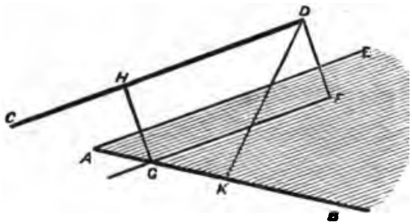
LINES IN SPACE.

THEOREM 19.

To two straight lines in space not being in the same plane one common perpendicular can be drawn, and this will be the shortest line joining the given lines.

Proof. Let AB and CD be the two straight lines, not in the same plane.

Through A draw a line AE parallel to CD; let fall a perpendicular DF to the plane AEB, and let the plane



CDF cut the plane AEB in the line FG, intersecting AB in G. Draw GH parallel to FD to meet CD in H, GH is the line required.

For since AE is parallel to CD, CD is parallel to the plane AEB, and therefore GF is parallel to CD. Therefore DF is perpendicular to CD.

Therefore also HG, which is parallel to DF, is perpendicular to CD and to the plane AEB,

and therefore HG is perpendicular to CD and AB.

Also if any other line DK be drawn to join the given lines,

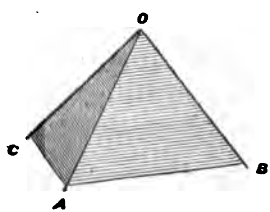
DK being oblique is $> DF$, (Th. 6, Cor. 1.)

and $DF = HG$,

therefore DK is $> HG$.

TRIHEDRAL ANGLES.

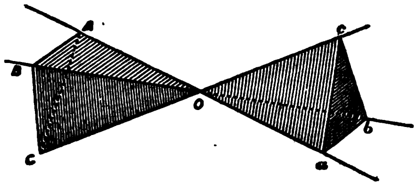
Def. 10. The figure formed by three planes meeting in one point is called a *trihedral angle.*



In the figure, the planes OAB, OAC, OBC, meeting at O, form the trihedral angle $OABC$.

It must be noticed that if the

edges of a trihedral angle are produced through the vertex, the pair of trihedral angles so produced are not super-

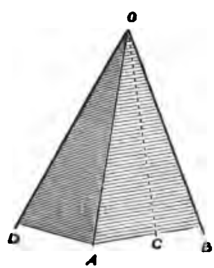


posable on one another, although the dihedral angles are equal, and the plane angles are equal, respectively.

For if the trihedral angle $OABC$ were placed on the trihedral angle $Oabc$, so that the dihedral angle OB coincided with the dihedral angle Ob, the face AOB would lie on the face cOb, and not on the face aOb. The angles are in fact right-handed and left-handed, and though corresponding and equal in every detail can no more be conceived as superposed on one another, so as to occupy the same space, than the form of the right hand on that of the left hand.

These angles were called *symmetrical* by Legendre.

Def. 11. The figure formed by more than three planes meeting in one point is called a *polyhedral angle.*



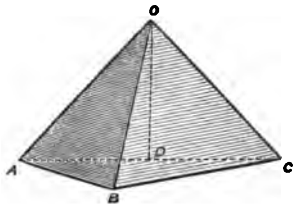
The plane angles are called the *faces* of the angle, and the intersections of the faces are called the *edges* of the angle.

A polyhedral angle is said to be *convex* when it lies wholly on one side of each of its faces.

THEOREM 20.

In any trihedral angle the sum of any two of its faces is greater than the third.

Proof. Let $OABC$ be a trihedral angle, and AOC the face which is not less than either of the others. It is required to prove that AOC is $< AOB + BOC$.



Make the angle AOD in the plane AOC equal to AOB.

Draw any straight line ADC, in the plane AOC, and make $OB = OD$, and join AB, BC.

Then in the two triangles AOB, AOD, AO, $OD = AO$, OB, and the included angles are equal, and therefore $AB = AD$.

But $$AB + BC > AC,$$

and therefore $$BC > DC;$$

and therefore in the triangles BOC, DOC, which have the sides BO, $OC = DO$, OC but the base $BC > DC$,

the angle $BOC > DOC$,

but $AOB = AOD$;

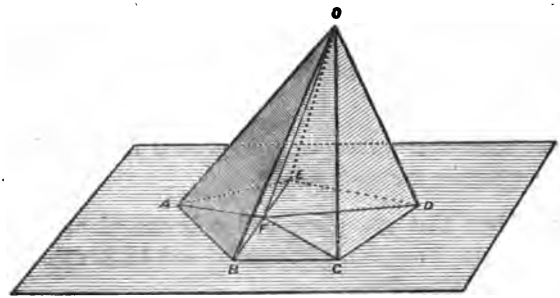
and therefore $$AOB + BOC > AOD + DOC$$
$$> AOC\text{[1]}.$$

THEOREM 21.

The sum of the faces of any convex polyhedral angle is less than four right angles[2].

[1] Euclid XI. 20. [2] Euclid XI. 21.

Proof. Let $OABCDE$ be a convex polyhedral angle; then will the sum of the plane angles at O be less than four right angles.



Let a plane be drawn to cut the edges on the same side of the vertex, in the points A, B, C, D, E.

Take any point F within the polygon $ABCDE$, and join OF, AF, BF, CF, DF, and EF.

Then since A is a trihedral angle,

therefore $$OAE + OAB > EAB, \qquad \text{(Th. 20.)}$$
$$> EAF + FAB,$$

and similarly for the other angular points of the polygon; therefore, by addition, the sum of the angles at the base of the triangles whose vertex is O is greater than the sum of the angles at the base of the triangles whose vertex is F;

but these two series of triangles are equal in number,

and therefore the sums of all their angles respectively are equal;

and therefore the sum of the angles at the vertex O is less than the sum of the angles at the vertex F;

but the angles at F are equal to four right angles,

and therefore the angles at O are less than four right angles.

Exercises on Section I.

1. Shew that of two obliques drawn to a plane from a given point, that oblique whose extremity is the more remote from the foot of the perpendicular is the greater, and conversely.

2. Shew that three planes in general intersect in one point: what are the exceptions?

3. Shew that one plane, and only one plane, can be drawn to pass through a given point and be perpendicular to a given straight line.

4. Find the locus of points equally distant from three given points.

5. If AB, BC, CD are lines, such that ABC, BCD are right angles, and AB is at right angles to the plane BCD, then will CD be at right angles to the plane ABC.

6. Given a plane MN, and two points P, Q on the same side of the plane, find a point A in the plane MN, such that $PA + AQ$ is a minimum.

7. From a given point, within or without a plane, to draw a normal to the plane.

SECTION II.

POLYHEDRA.

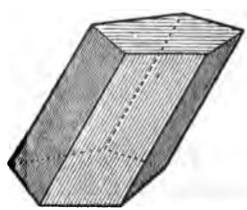
Def. 12. A *polyhedron* is a figure bounded on all sides by planes.

These planes determine by their intersection the *faces*, *edges*, and *angles* of the polyhedron.

Def. 13. A polyhedron is *convex* when it lies wholly on one side of each of its faces.

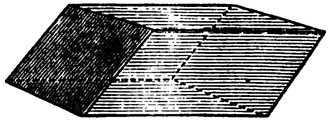
Def. 14. A polyhedron is *regular* when its faces are equal and regular polygons, and its angles are equal.

Def. 15. A *prism* is a polyhedron, whose sides are parallelograms, and extremities equal polygons in parallel planes.



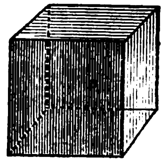
The intersections of the parallelograms are called its edges, and the polygons are called its bases.

Def. 16. A *parallelepiped* is a polyhedron bounded by three pairs of parallel planes.

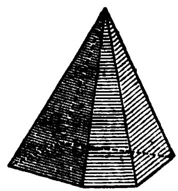


Def. 17. A prism is called *right* or *oblique* according as its edges are perpendicular or oblique to its base.

Def. 18. A *cube* is a rectangular prism on a square base, whose height is equal to the side of its base; it is therefore bounded by six equal squares, the three edges meeting in any point being at right angles to one another.

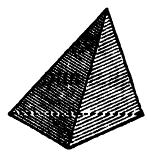


Def. 19. A *pyramid* is a polyhedron, one of whose faces is a polygon, and the others are triangles, whose bases are the sides of the polygon, and which have as a common vertex any point not in the plane of the polygon.



The common vertex is called the *vertex* of the pyramid, the polygon its *base*, and the perpendicular from the vertex to the base is called its *altitude*.

Def. 20. A *tetrahedron* is a pyramid on a triangular base.

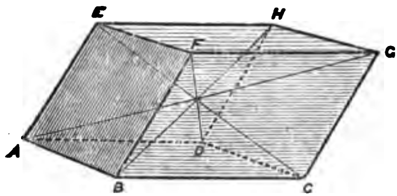


THE PARALLELEPIPED.

THEOREM 22.

The opposite faces of a parallelepiped are parallelograms, and are equal and parallel, and its diagonals pass through one point and bisect one another.

Let $ABCG$ be a parallelepiped, that is, let its opposite faces be parallel planes.



Then since the lines of intersection of parallel planes are themselves parallel (Th. 13); therefore, AB, EF, HG, DC are all parallel; and similarly AE, BF, CG, DH are parallel, and likewise EH, FG, BC, AD.

Therefore each of the faces is a parallelogram. And because AB, BF are respectively parallel to DC, CG,

therefore the angle $ABF =$ the angle DCG. (Th. 9.)

But AB, $BF = DC$, CG, being opposite sides of parallelograms;

therefore the opposite faces $ABFE$, $DCGH$ are equal and parallel.

And in the same manner it may be proved that $EFGH$, $ABCD$ are equal and parallel; and also $ADHE$, $BCGF$ are equal and parallel.

Again, since FG, AD are parallel, a plane can pass through them both; and since $AD = FG$, the figure $AFGD$ is a parallelogram.

Therefore AG and FD bisect one another.

Similarly EC and FD, or BH and FD bisect one

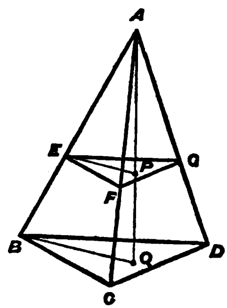
another, that is, AG, EC, FD, BH have a common point of bisection.

COR. *Hence a parallelepiped is a prism on a parallelogram as base.*

THE PYRAMID.

THEOREM 23.

The areas of the sections of a pyramid made by planes parallel to the base are proportional to the squares of their distances from the vertex.



Let $ABCD$ be a pyramid on a triangular base BCD, and let EFG be a section parallel to the base.

Draw APQ perpendicular to the base, meeting the parallel planes in P, Q.

Then will $EFG : BCD :: AP^2 : AQ^2$.

Join EP, BQ.

Because EFG is parallel to BCD, and they are cut by the plane ABC; therefore EF is parallel to BC. (Th. 13.)

Similarly EG is parallel to BD, and FG to CD; and therefore the triangle EFG is equiangular to the triangle BCD, (Th. 9.)

and therefore the triangles are similar.

And the triangle AEF is equiangular to the triangle ABC, and AEP to ABQ.

Therefore

the area EFG : the area $BCD :: EF^2 : BC^2$ (Bk. III.)

$$:: AE^2 : AB^2$$

$$:: AP^2 : AQ^2.$$

And if the pyramid is on a polygonal base, it can be decomposed into pyramids on triangular bases, and the theorem proved in the same manner.

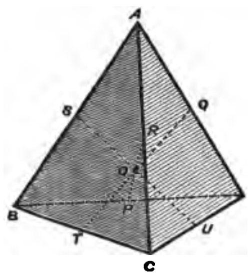
COR. *Hence if two pyramids are on equal bases and of equal altitude, the sections of them made at equal distances from the base are equal.*

THE TETRAHEDRON.

THEOREM 24.

The planes which contain each an edge of a tetrahedron and bisect the opposite edge, pass through one point.

Let $ABCD$ be a tetrahedron, and let its sides be bisected in P, Q, R, S, T, U.



Then since SQ and TU are each of them parallel to BD and half of BD,

therefore $SQUT$ is a parallelogram,

and therefore its diagonals bisect one another.

Similarly $SRUP$, $RQPT$ are parallelograms.

Hence the lines SU, TQ, RP bisect one another in one point O.

But the plane BCQ plainly contains TQ, and therefore it passes through O.

Similarly all the planes like BCQ pass through O.

The Regular Polyhedra.

/Theorem 25.

There cannot be more than five regular Polyhedra.

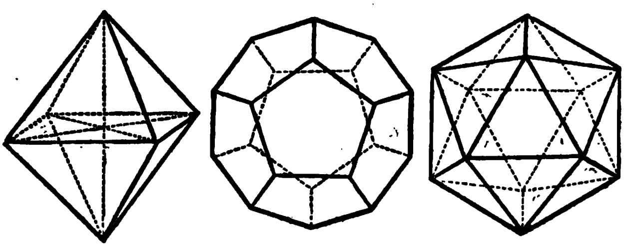
Since three faces at least must meet to form each solid angle, and since the sum of the plane angles at each vertex is less than four right angles, it is plain that the faces of a regular polyhedron must be equilateral triangles, squares, or regular pentagons; for three hexagons placed with an angular point in common, and having edges common, would make the adjacent angles equal to four right angles.

Similarly four squares, or six equilateral triangles, would give a solid angle whose plane faces equal four right angles.

The only possible polyhedra therefore will have their solid angles formed respectively

(1) by three equilateral triangles,
(2) by three squares,
(3) by four equilateral triangles,
(4) by three pentagons,
(5) by five equilateral triangles.

These solids are known as the *tetrahedron*, *cube*, *octahedron*, *dodecahedron*, and *icosahedron* respectively.



EULER'S THEOREM.

THEOREM 26.

If E *is the number of edges,* F *the number of faces,* V *the number of vertices of any Polyhedron, then* $E+2=F+V$.

Consider a single polygonal face, with m edges and m vertices, and let the polyhedron be conceived as constructed by adding to this face other faces in succession; and let E', F', V' be, at any stage of the process, the number of edges, faces, and vertices then constructed.

Therefore when $F'=1$, $E'=m$, $V'=m$, and $\therefore E'-V'=0$.

Now add one more face, having one common edge, and two common vertices, with the former;

thus we add one more new edge than new vertex,

therefore when $F'=2$, $E'-V'=1$.

And it may easily be seen that until the figure is closed we continually add for each new face one more new edge than new vertex;

so when $F'=3$, $E'-V'=2$;

and when $F'=F-1$, $E'-V'=F-2$.

The polyhedron is now closed with the exception of one face: but, by the addition of this, we add no new edges or new vertices, and therefore, when $F'=F$, $E-V=E'-V'$.

and therefore $E-V=F-2$,

or, $E+2=F+V$[1].

[1] This proof is due to Cauchy.

Exercises on Polyhedra.

1. Find the number of edges in a pyramid, and in a prism, on a polygon of n sides as base.

2. Into how many portions do three planes divide space?

3. Into how many portions do four planes, not passing through one point, divide space? Ans. 15.

4. If a tetrahedron is cut by a plane parallel to two opposite edges, the section will be a parallelogram. Find the position in which this section will be a maximum.

5. Prove that the three planes which pass through the edges of a trihedral angle, and are perpendicular to the opposite faces of that angle, pass through one point.

6. Prove that the six planes which bisect the six internal dihedral angles of a tetrahedron pass through one point.

7. To cut a solid tetrahedral angle by a plane, so that the section shall be a parallelogram.

Let the opposite faces intersect in two lines α, β. Then any plane parallel to α and β will satisfy the conditions.

8. The edges of a rectangular parallelepiped are a, b, c, a being the greatest, and c the least. Shew that of the three diametral planes the area of the largest is $a\sqrt{b^2+c^2}$, of the least $c\sqrt{a^2+b^2}$.

9. Verify Euler's Theorem in the case of all the regular polyhedra.

SECTION III.

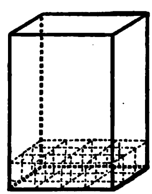
STEREOMETRY.

This part of Geometry treats of the volumes of Solids and their numerical values.

The *unit of volume* is the cube whose edge is the unit of length.

Def. 21. The volume of a solid is the number of units of volume it contains.

The volume of any right prism is plainly doubled when its height is doubled, and increased in whatever proportion its height is increased; and similarly it is doubled when the area of its base is doubled, and increased in whatever proportion its base is increased. And if its base contains m square units, and its height h units, its volume will contain hm units or $= hm$.



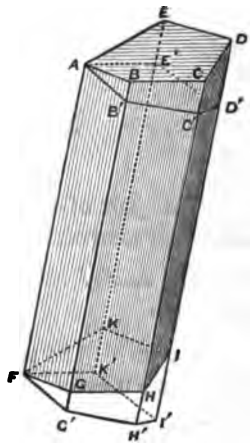
Therefore generally if h, m be fractional or incommensurable, the volume of a right prism, whose base is m and height is h, is mh.

It is obvious that if the prism is a rectangular parallelepiped whose edges are a, b, c, its volume $= abc$.

Therefore the volume of a cube whose edge is $a = a^3$.

Theorem 27.

The volume of an oblique prism is equal to the area of its right section multiplied by its length.



Let $ABCDEFGHIK$ be an oblique prism, and let a section at right angles to its edges be made by the plane $AB'C'D'E'$.

Then will the volume of the prism be equal to the area of the polygon $AB'C'D'E'$ multiplied by AF.

Conceive the wedge-shaped solid $ABCDEB'C'D'E'$ placed at the other extremity of the prism, A falling on F, and B on G, &c.

Then the volume of the original prism will be equal to that of the right prism $AB'C'D'E'FG'H'I'K'$.

But the volume of this, being a right prism, is equal to the area of its base into its altitude;

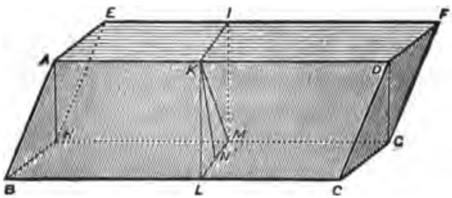
therefore the volume of the oblique prism is equal to the area of its right section into its edge.

Theorem 28.

The volume of a parallelepiped is equal to the product of its base into its altitude.

Let $ABCDEFGH$ be a parallelepiped.

Let a plane $IKLM$ cut the edges AD, BC, &c. at right angles.



Then by the preceding theorem the volume of the parallelepiped = area $IKLM \times BC$.

Draw KN perpendicular to LM, so that KN = the altitude of the parallelepiped.

Then area $IKLM = KN \times LM$, (Bk. I.)

therefore volume of figure $= KN \times LM \times BC$. (Th. 27.)

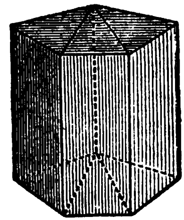
But $LM \times BC$ = area $BCGH$,

therefore volume of figure $= KN \times$ area $BCGH$

= altitude × base.

COR. I. *The volume of an oblique triangular prism, whose bases are* AHE, DGF, *would equal area of triangle* IKM × BC (Th. 27), *and would therefore equal half the volume of the parallelepiped, which equals* IKLM × BC; *but the volume of the parallelepiped = base* ABHE × *altitude, by the present Theorem; and the base of the prism* AEH *being half the base* ABHE, *the volume also of the prism equals its base multiplied by its altitude.*

It must be noticed that the prisms into which a plane $ADHG$ divides the parallelepiped are not *superposable*, though they are thus proved to be *equivalent* in volume.



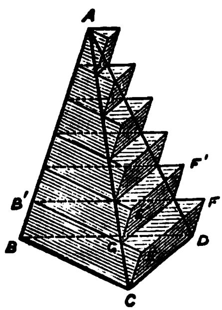
COR. 2. *The volume of any oblique prism is equal to its base multiplied by its altitude.*

For it may be divided into triangular prisms whose altitude is the same, and the sum of whose bases equals the base of the original prism.

THE VOLUME OF THE PYRAMID.

THEOREM 29.

The volume of a pyramid is one-third of its base × its altitude[1].



Let $ABCD$ be a triangular pyramid.

Divide AC into any number of equal parts, and through the points of section draw planes parallel to the base BCD, and through CD, and through the intersections of these planes with ACD, draw planes parallel to AB, as in the figure.

Then the volume of the pyramid is less than the sum of the volumes of the prisms BEF, $B'E'F'$, &c., by the prismoidal solids $GECDF$, &c.

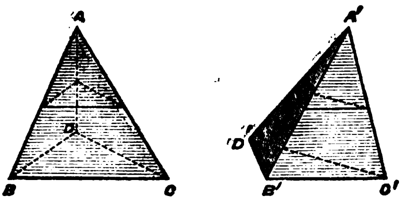
But the sum of all these is less than the prism BEF,

[1] Euclid XII. 7.
This theorem requires a new method of reasoning. It is said to have been first solved by Eudoxus, a contemporary of Plato.

which may be made as small as we please by dividing AC into a sufficiently great number of parts.

Therefore the volume of the pyramid is equal to the limit of the sum of the prisms when their number is made indefinitely great.

Again, if two pyramids $ABCD$, $A'B'C'D'$ are on equal bases and of equal altitude, since planes at equal distances from the base would make equal sections (Th. 23), it follows that the prisms formed as described above in two such pyramids would be respectively equal, and therefore the sums of the prisms would be equal; and therefore pyramids of equal bases and equal altitudes are equal.



NOTE. The proof that pyramids of equal bases and altitudes are equal is somewhat difficult fully to grasp. It may be regarded thus. Let V, V' be the volumes of the two pyramids; let them be cut by planes as in the theorem, and let the sums of the prisms be S, S'.

Then it is proved (1) that $S-V$, and $S'-V'$, may be made as small as we please;
and (2) that $S=S'$,
from which we infer that $V=V'$,
for if $V-V'=a$,
then $S-V$ and $S'-V'$ would differ by a;
and therefore could not both be made as small as we please.

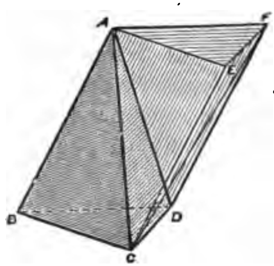
Hence if $ABCD$ is a pyramid, through C, D draw lines parallel to AB, and cut them by a plane AEF parallel to BCD, forming a prism $ABCDEF$.

Join CF.

Then the prism is divided into three pyramids, $ABCD$, $CEAF$, $FDCA$.

But of these $ABCD = CEAF$, since they are on equal bases BCD, AEF, and have the same altitudes.

Also $ABCD = FDCA$ being on equal bases ABD, FAD, and having the same altitude, therefore each pyramid



$= \frac{1}{3}$ prism, therefore the volume of a triangular pyramid $= \frac{1}{3}$ base $\times$ altitude.

COR. *The volume of any pyramid is equal to* $\frac{1}{3}$ *base* $\times$ *altitude.*

Def. 22. A *frustum* of a pyramid is the volume included between two parallel planes.

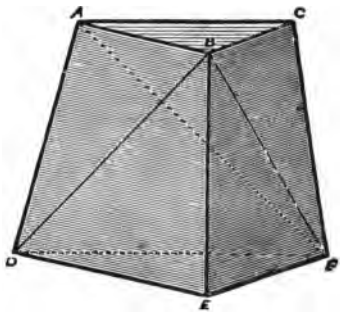
Def. 23. A *frustum* of a prism is the volume included between two planes not parallel.

THEOREM 30.

To find the volume of a frustum of a triangular pyramid, in terms of its altitude, and the areas of its bases.

Let $ABCDEF$ be a frustrum of a pyramid, the plane ABC being therefore parallel to DEF; it is required to find its volume.

Let it be cut by planes BDF, BAF into three pyramids, viz. $BDEF$, $FABC$, $BADF$. Let the areas of ABC,



DEF be b, B, and the altitude of the frustum of the pyramid h.

Then $BDEF = \frac{1}{3} hB$, and $FABC = \frac{1}{3} hb$, and it remains only to express the volume of the pyramid $BADF$.

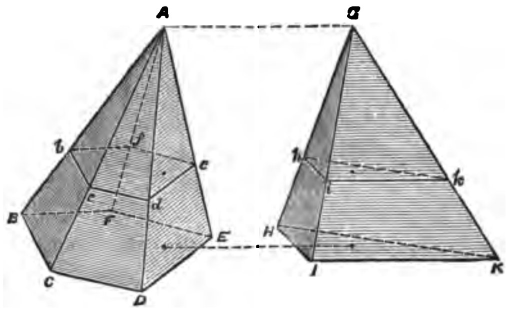
$$\text{Now } BADF : BACF :: \text{base } ADF : \text{base } ACF$$
$$:: DF : AC$$
$$:: \sqrt{B} : \sqrt{b},$$
$$\text{and } \therefore BADF = \frac{\sqrt{B}}{\sqrt{b}} \times BACF$$
$$= \frac{\sqrt{B}}{\sqrt{b}} \cdot \frac{1}{3} hb = \frac{1}{3} h \sqrt{Bb};$$

$$\therefore \text{ volume of truncated pyramid} = \frac{1}{3} h \{B + \sqrt{Bb} + b\}.$$

COR. This may be expressed as follows;

The volume of the frustum of a triangular pyramid is equal to the sum of the volumes of three pyramids, whose height is the height of the frustum, and whose bases are respectively the parallel faces of the frustum, and the geometrical mean between them.

This property may be extended to the frusta of pyramids in general as follows:



Let $bcdefBCDEF$ be a frustum of a polygonal pyramid $ABCDEF$.

Let B, b, h be the areas of its ends, and its altitude.

Construct a triangle HIK equal in area to the polygon $BCDEF$, and let a pyramid $GHIK$ of the same altitude as the former pyramid be constructed on it; and let the pyramids be placed on the same plane, and let the section of the latter by the plane $bcdef$ be the triangle hik, and let its area be therefore b (Th. 23, Cor.); and the altitude of the frustum will be h.

Then since pyramids of equal bases and equal altitudes are equal, the pyramids ABE, Abe are respectively equal to the pyramids $GHIK$, $Ghik$.

Therefore the frustum bE = the frustum hK.

Also since the areas of parallel sections of the pyramids are to one another as the squares of their distances from the vertex,

$$\therefore \text{polygon } be : \text{polygon } BE :: hik : HIK,$$

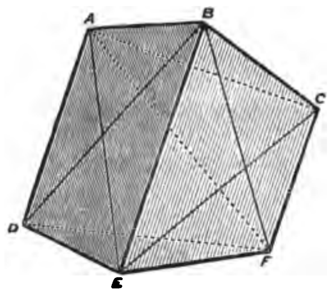
$$\text{but } HIK = BE = B,$$

$$\therefore hik = be = b;$$

and $\therefore$ volume of frustum $= \frac{1}{3} h (B + \sqrt{Bb} + b)$.

THEOREM 31.

To find the volume of the frustum of a prism.



Let $ABCDEF$ be the frustum.

above Let the base $DEF = b$, and let h, h', h'' be the altitudes of A, B, C ~~of~~ the plane ~~above A, B, C,~~ DEF.

First, cut it by a plane BDF, detaching a pyramid, whose base is DEF, and height the altitude of B above DEF;

$$\therefore \text{its volume} = \frac{1}{3} bh'.$$

Divide the remainder by a plane BAF into two pyramids $BADF$, $BACF$.

But, remembering that the edges of the prism are parallel to one another, and that pyramids on the same base and of equal altitudes are equal, it will be seen that

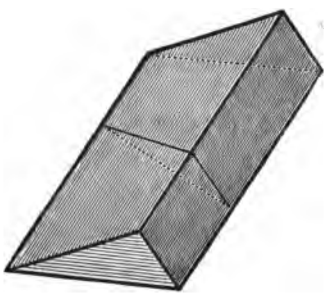
$$BADF = EADF = \frac{1}{3} bh,$$

also
$$ABCF = DBCF = DECF$$
$$= \frac{1}{3} bh'';$$

$$\therefore \text{ volume of frustum} = \frac{1}{3} b(h + h' + h'').$$

Cor. 1. *In a right prism the volume of the frustum = area of right section multiplied by one-third of the sum of the parallel edges.*

Cor. 2. *The volume of a frustum of an oblique prism = the area of its right section* $\times \frac{1}{3}$ *sum of its parallel edges.*



For it may be divided into two frusta of a right prism by a plane at right angles to its edges, and if s, s' are

the sum of their edges taken separately, and a the area of the right section, their volumes are

$$\frac{1}{3}as + \frac{1}{3}as',$$

and therefore the total volume

$$= \frac{1}{3}a(s + s'),$$

but $s + s'$ equals the sum of the parallel edges of the frustrum; and therefore volume of frustrum is equal to

$$\text{area of right section} \times \frac{1}{3} \text{ sum of the parallel edges.}$$

THE CYLINDER.

Def. 24. *A cylindrical surface* in general is produced by a line which moves always parallel to itself, and intersects a given curve in space.

Def. 25. *A right circular cylinder* is the solid produced by the revolution of a rectangle round one of its sides.

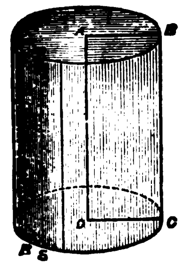
Thus, let $ABCD$ be a rectangle revolving round AD, then BC will trace out the lateral surface of the cylinder.

AB and DC will trace out circles which are called the *bases* of the cylinder. AD is called the *axis* of the cylinder.

It is plain that any section of a right circular cylinder parallel to the base will be a circle.

A section made by a plane parallel to the axis will in general consist of two parallel straight lines.

Thus if $PQRS$ be a plane, it will cut the cylinder in the two parallel lines PR, QS.



When these two lines approach each other and coincide, the plane will then *touch* the cylinder along one of its generating lines.

All other sections of the circular cylinder, made by planes inclined to the axis, will be oval curves, called *ellipses*.

Theorem 32.

To find the lateral surface and volume of a cylinder.

Inscribe in the cylinder a polygonal prism, and let one of its faces be the parallelogram $PQSR$.

Then, since

$$PQSR = PR \times RS,$$

the surface of the prism = height of cylinder × circumference of the polygonal base of prism, but when the number of faces of the prism is indefinitely increased, and their size indefinitely diminished, the circumference of the polygon has for its limit the circumference of the circle, (Bk. II.)

and therefore the lateral surface of the prism has for its limit the lateral surface of the cylinder,

therefore the lateral surface of the cylinder equals the height of the cylinder multiplied by the circumference of the cylinder.

Similarly, the limit of the volume of the prism is equal to the volume of the cylinder, and the limit of the base of the prism is the base of the cylinder.

But the volume of the prism = its height × its base,

and ∴ the volume of the cylinder = its height × its base.

NOTE. The surface of a cylinder may be conceived as unrolled and laid on a plane, and will then form a rectangle.

Surfaces which, like the cylindrical surface, may be conceived as laid on a plane without tearing them, are called *developable* surfaces.

If h is the height of the cylinder, r the radius of the base, and therefore $2\pi r$ the circumference of the base, it follows that the lateral surface of the cylinder $= 2\pi rh$, and the total surface

$$= 2\pi rh + 2\pi r^2$$

$$= 2\pi r(r+h),$$

and the volume of the cylinder

$$= \pi r^2 h.$$

THE CONE.

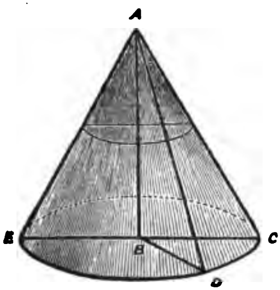
Def. 26. *A conical surface in general* is produced by a straight line constrained to pass through one fixed point, and to intersect a curve in space.

Def. 27. *A right circular cone* is the solid produced by the revolution of a right-angled triangle round one of the sides containing the right angle.

Thus let ABC be a triangle right-angled at B, and let it revolve round AB.

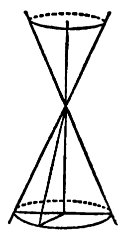
Then AC will trace out a conical surface.

BC will trace out a circle, which is called the base of the cone.



A is called the *vertex*, AC is called the *slant side*, AB the *altitude* or the *axis* of the cone.

It is plain that any point in AC describes a circle as the line AC rotates round AB, and therefore that all sections of a right circular cone, parallel to the base, are circles.



It must be observed that the generating lines are considered to be indefinitely produced both ways, and that the total *conical surface* includes two cones whose vertices are at the same point.

A *conic section* is the trace on this surface made by a plane which intersects it.

The plane may have different positions relative to the cone, and thus produces different conic sections.

(1) Let the plane pass through the vertex, and not meet the surface elsewhere. Then the conic section is a *point*.

(2) Let the plane pass through the vertex and cut the cone. Then the conic section will consist of two straight lines, passing through the vertex.

(3) Let the plane pass through the vertex and touch the cone along one of its generating lines. Then the conic section consists of one straight line, which can [redacted] be regarded as two coincident straight lines.

(4) Let the plane not pass through the vertex, and be at right angles to the axis. Then the conic section will be a circle.

(5) Let the plane cut all the generators on the same side of the vertex. Then the section is called an *ellipse.*

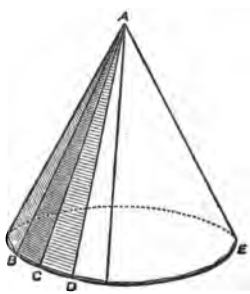
(6) Let the plane be parallel to one of the generators, and thus consist of one infinite branch. Then the section is called a *parabola.*

(7) Let the plane cut the cone on both sides of the vertex, and thus have two infinite branches. Then the section is called a *hyperbola.*

THEOREM 33.

To find the lateral surface and volume of a right circular cone.

Let $ABCDE$ be a right circular cone.



Inscribe in its base a regular polygon BCD... and let CD be one of its sides. Let planes ACD... be drawn. Then there is inscribed in the cone a polygonal pyramid; and when the number of the sides of the polygon is indefinitely increased, the

limits of the lateral surface and volume of the pyramid are equal to the lateral surface and volume of the cone.

But area of triangle $ACD = \frac{1}{2} CD \times$ perpendicular from A on CD.

And therefore lateral surface of pyramid $= \frac{1}{2}$ circumference of polygon $\times$ perpendicular from A on one of its sides.

And in the limit the circumference of polygon = circumference of circle, and the perpendicular from A on $CD = AC$.

And therefore the lateral surface of cone $= \frac{1}{2}$ circumference of its base $\times$ slant side of cone.

Similarly, since the volume of the pyramid $= \frac{1}{3}$ altitude $\times$ base, and the limit of its base is the base of the cone, and the limit of its volume the volume of the cone;

$\therefore$ the volume of the cone $= \frac{1}{3}$ altitude of the cone $\times$ its base[1].

COR. 1. If h be the height, r the radius of base, a the slant side of the cone, the circumference of the base $= 2\pi r$, and area of base $= \pi r^2$.

And therefore the lateral surface of the cone

$$= \frac{1}{2} \times 2\pi r \times a = \pi ra.$$

And the total surface

$$= \pi ra + \pi r^2 = \pi r(a + r).$$

[1] Euclid XII. 10.

And volume of cone

$$=\frac{1}{3}\pi r^2 h$$

$=\frac{1}{3}$ *vol. of cylinder on the same base and having the same altitude.* (Th. 32.)

Also $$a^2=h^2+r^2.$$

COR. 2. The volume of the frustum of a cone made by a plane parallel to its base may be deduced from that of the frustum of a pyramid. See Theorem 30, Cor.

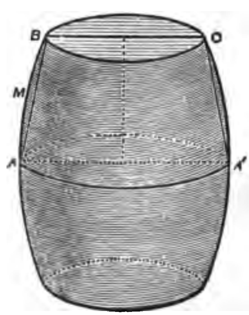
If r_1, r_2 are the radii of the circular ends, h the height of the frustum, this volume (v) becomes

$$\frac{1}{3}\pi h\,(r_1^2+r_1r_2+r_2^2).$$

This formula may be put into the form

$$v=\pi h\left\{\left(\frac{r_1+r_2}{2}\right)^2+\frac{1}{3}\cdot\left(\frac{r_1-r_2}{2}\right)^2\right\}.$$

COR. 3. The volume of a cask may be gauged by the aid of this formula.



Let R, r be the radii of its greatest section and its extremity; h its height.

Then its volume is nearly equal to twice that of the frustum of a cone $AA'CB$.

And therefore its volume

$$=\frac{1}{3}\pi h\,(R^2+Rr+r^2) \text{ nearly.} \qquad (1)$$

This neglects the small volume generated by the segment AMB, and therefore the volume given is too small.

By writing R^2 for Rr, we get the formula

$$\frac{1}{3}\pi h\,(2R^2+r^2), \qquad (2)$$

which is a little too large.

Ex. To find the number of gallons in a cask whose height is 3 feet, and greatest and least circumferences are 10 and 8 feet respectively, a gallon containing 277·274 cubic inches.

The volume will be found by (1) and (2) to be greater than 121, and less than 131 gallons.

The nearest approximation in the case of most casks is said to be given by the formula

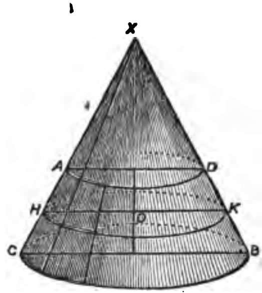
$$V=\frac{1}{3}\pi h\left\{2R^2+r^2-\frac{1}{3}\,(R^2-r^2)\right\}.$$

This would give 125 gallons as the contents of the cask above.

COR. 4. *The lateral surface of the frustum of a cone may be found by multiplying its slant side by the circumference of the circular section equidistant from its parallel faces.*

Let $ACBD$ be the frustum, HK equidistant from AD and BC.

Then the surface may be regarded as the limit of the sum of a number of trapezoids, whose sides are the sides of the frustum, and parallel ends the sides of regular polygons inscribed in the circular ends.



But the area of a trapezold $= \frac{1}{2}$ altitude × (sum of parallel sides).

And therefore the sum of the surfaces of the trapezolds $= \frac{1}{2}$ altitude × sum of the circumferences of its polygonal ends.

And therefore the lateral surface of the frustum $= \frac{1}{2}$ slant side × sum of circumferences of the circular ends.

But $\frac{1}{2}$ sum of the circumferences of its ends = circumference of the circle equidistant from the ends.

And therefore the lateral surface of the frustum $= AC$ × circumference of HK

$$= 2\pi \, . \, AC \, . \, HO.$$

Def. 28. *Similar polyhedra* are such as have all their polyhedral angles equal, each to each, and are contained by the same number of similar faces.

Similar polyhedra [illegible] home [illegible]

THEOREM 34.

Similar polyhedra are to one another in the ratio of the cubes of their corresponding edges.

Let P, p be the polyhedra; and let them be divided into the same number of similar pyramids, by joining their vertices to two points correspondingly situated, one in each polyhedron.

And let the volumes of the pyramids be $A, B, C \ldots a, b, c$ respectively, and let F, f, the bases of A, a, be corresponding faces, E, e corresponding edges of those faces, and H, h altitudes of the pyramids on those faces. Then

$$A = \frac{1}{3} HF, \quad a = \frac{1}{3} hf.$$

Therefore $\quad A : a :: HF : hf,$

but $\quad H : h :: E : e,$

and $\quad F : f :: E^2 : e^2,$ (Bk. III.)

and therefore $\quad A : a :: E^3 : e^3.$

Similarly $\quad B : b :: E^3 : e^3,$

and $\quad C : c :: E^3 : e^3,$

and therefore

$$A + B + C + \ldots : a + b + c + \ldots :: E^3 : e^3,$$

that is, $\quad P : p :: E^3 : e^3.$

COR. Since any similar solids may be considered as the limiting form of similar polyhedra, when the number of faces is indefinitely increased, it follows that *similar solids have to one another the ratio of the cubes of their linear dimensions.*

EXERCISES ON SECTION III.

1. Find the number of cubic feet in the trunk of a tree, 70 feet long, the diameters of its ends being 10 and 7 feet.

2. Find the content of a right-angled cone 1 foot high.

3. A mound of earth is raised with plane sloping sides: the dimensions at the bottom are 80 yards by 10, at the top 70 by 1, and it is 5 yards high; find its cubical content.

4. Find the content by all the formulæ in Th. 33, Cor. 3, of a cask 4·6 ft. high, and measuring 13 and 10 feet in its greatest and least circumference.

5. A bath 6 feet deep is excavated, the area of the surface at the top is 100 square yards, at the bottom 81 square yards. Find the number of gallons of water it will hold.

6. A railway embankment across a valley has the following measures. Width at top 20 feet, at base 45 feet, height 11 feet, length at top 1020 yards, at base 960 yards. Find its volume.

7. How many cones can be described to contain three straight lines which intersect one another but are not in the same plane?

8. What is the locus of points at a given distance from a given finite straight line?

9. To bisect the lateral surface of a right cone by a plane parallel to the base.

10. To divide in any required ratio the lateral surface of a cone by a plane parallel to the base.

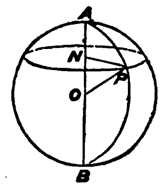
SECTION IV.

THE SPHERE.

Def. 29. The *Sphere* is the solid produced by the revolution of a semicircle about the diameter.

Let APB be the semicircle revolving round AB the diameter.

Let AB be bisected in O.



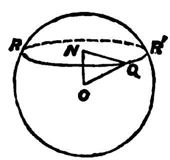
Then it follows from the definition:

(1) That P is always at the same distance from O. The fixed distance is called the *radius* of the sphere.

(2) That every point P in the semicircle describes a circle whose plane is at right angles to AB, and centre lies in AB.

THEOREM 35.

Every section of a sphere made by a plane is a circle.



Let a plane RQR' cut the sphere whose centre is O; then shall the section be a circle.

Draw ON perpendicular to the plane RQR', meeting it in N, and join OQ.

Then since OQ is an oblique of constant length, being the radius of the sphere,

therefore its extremity Q is at a constant distance from N the foot of the perpendicular; (Th. 6, Cor. 1.)

that is, the locus of Q is a circle whose centre is N.

COR. 1. *If the plane of section passes through the centre, the section is a circle whose radius is equal to the radius of the sphere.*

For in this case O coincides with N.

This circle is called a *great circle of the sphere.*

All other circles are called *small* circles.

COR. 2. If ON produced meets the surface in A, B, every point in the circle RQR' is equally distant from A, B.

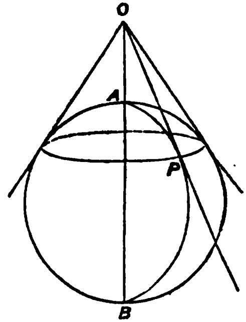
The points A, B are called the *poles* of the circle RQR'.

COR. 3. *A tangent plane is at right angles to the radius through its extremity.*

For if the plane RQR' moves parallel to itself, the centre of the circle RQR' will always lie in the line NO; ultimately, therefore, when N has moved up to the surface, the circle will have become indefinitely small, and the plane has only one point in common with the sphere. The plane is then said to touch the sphere at that point.

THEOREM 36.

A right circular cone may be circumscribed to a sphere, and will touch it along the circumference of a circle, whose plane is perpendicular to the axis of the cone.



Let APB be the semicircle by whose revolution round the diameter AB the sphere is generated; and let OP be the tangent at P, meeting BA produced in O.

Then as the semicircle and line OP revolve round OAB, OP will touch the sphere along the path of P.

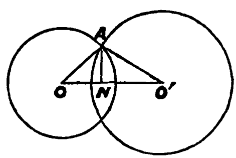
But the point P traces out a circle in a plane at right angles to AB; and OP therefore describes a right circular cone. (Def. 27.)

Therefore the sphere touches the cone along the circumference of a circle whose plane is perpendicular to the axis of the cone.

COR. *Tangents drawn to a sphere from an external point are equal.*

THEOREM 37.

The line of intersection of two spheres is a circle, whose plane is at right angles to the line joining their centres.



Let O, O' be the centres of two intersecting spheres; A any point in their line of intersection.

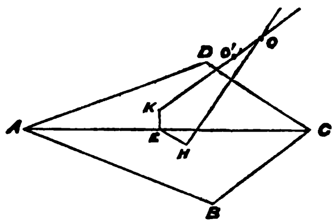
Draw AN perpendicular to OO'.

Then since the triangle OAO' is constant, AN is constant and meets OO' in a fixed point.

Therefore the locus of A is a circle whose centre is N and radius NA, and whose plane is at right angles to the line OO'.

THEOREM 38.

Through four points not in the same plane, one sphere and only one can be described.



Let A, B, C, D be the four points, and let H, K be the centres of the circumscribed circles of the triangles ABC, ADC.

Draw HO, KO' perpendiculars to the planes ABC, ADC. Then these lines will intersect in some point O.

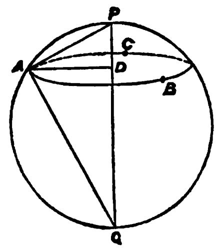
For since the plane ABC is not identical with the plane ADC, the perpendiculars to these planes are not parallel; and since HO is the locus of points equidistant from A, B, C, and KO' the locus of points equidistant from A, D, C, then HO and KO' are both equidistant from A and C; and therefore HO and KO' both lie in the plane which bisects AC at right angles. (Th. 6, Cor. 2.)

Therefore HO and KO' intersect in some point O, which will be therefore equidistant from A, B, C, D.

Therefore O is the centre of a sphere which passes through A, B, C, D.

PROBLEM.

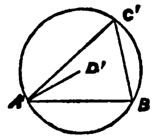
Given a portion of a spherical surface, to find, by the aid of ruler and compasses, the radius of the surface.



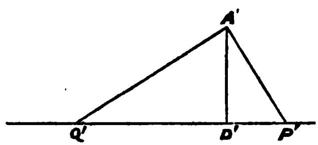
Construction. With any point P as centre describe a small circle ABC on the given portion of the surface of the sphere. (Th. 33, Cor. 2.)

Then the diameter will be the line through P at right angles to the plane of this circle, and subtending a right angle at A.

Take, therefore, any three points A, B, C on this circle; and by the aid of the compasses measure the distances AB, BC, CA, and describe a triangle $A'B'C'$ having its sides equal to AB, BC, CA:



Describe its circumscribing circle, and determine its radius $A'D'$.



Draw through D' a line at right angles to $A'D'$, and with centre A', and radius $= AP$, describe a circle to cut it in P'; join $A'P'$, and through A' draw $A'Q'$ at right angles to AP' to meet $D'P'$ in Q', then $P'Q'$ is the diameter of the sphere.

Proof. For if PQ were the diameter of the sphere it would pass through D the centre of the circle ABC, and be at right angles to its plane, and subtend a right angle at A, a point in the semicircle PAQ.

But by construction the triangles ADP, $A'D'P'$ are equal,

and therefore also the triangles APQ, $A'P'Q'$ are equal;
and therefore $P'Q' = PQ$,
for $P'Q'$ is the length of the diameter of the sphere.

THEOREM 39.

To find the surface of a sphere.

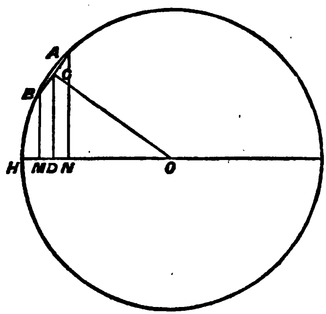
Let ABH be the semicircle, AB a small arc of it; draw AB the chord.

Then the zone of the sphere generated by a small arc AB is ultimately equal to the surface of the frustum of a cone, whose slant side is AB and axis HO, when the chord AB is diminished without limit; and the surface of the sphere is the sum of all such zones.

Bisect AB in C, and draw CD perpendicular to HO.

The surface of the frustum

$$= 2\pi \,.\, CD \times AB. \qquad \text{(Th. 33, Cor. 4.)}$$



Join CO, and let MN be the projection of AB on HO; then, by similar triangles,

$$CD : CO :: MN : AB,$$

and $$\therefore\; CD \times AB = CO \times MN;$$

therefore surface of frustum

$$= 2\pi\ CO \times MN.$$

But in the limit $CO = \text{radius} = r$;

$\therefore$ surface of zone has for its limit $2\pi r \,.\, MN$;

but if a number of chords, occupying the whole semi-circumference, were drawn, their projections MN... would together make up the diameter $2r$:

and $\therefore$ surface of sphere = sum of all the zones

$$= 2\pi r \times 2r$$
$$= 4\pi r^2.$$

Cor. 1. *The area of any zone is in proportion to its height alone.*

For the area of the zone described by the revolution of any arc AB

$$= 2\pi r \,.\, MN.$$

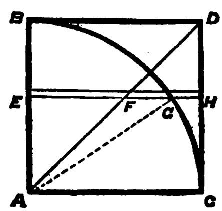
Cor. 2. If a cylinder were circumscribed about the sphere, the areas of its ends together $= 2\pi r^2$, and of its curved surface $= 2\pi r \times 2r$ (Th. 32) $= 4\pi r^2$;

$\therefore$ total surface of circumscribing cylinder $= 6\pi r^2$;

$\therefore$ *surface of sphere* $= \frac{2}{3}$ *surface of the circumscribing cylinder.*

Theorem 40.

The volume of a sphere $= \frac{2}{3}$ *of that of the circumscribing cylinder.*



Let ABC be the quadrant of a circle, BD, DC tangents at its extremities; join AD, and conceive the figure to rotate round AB.

Then ABD will generate a cone, ABC will generate a hemisphere, and $ABDC$ will generate a cylinder.

Let parallel planes, very near together, cut the figures at right angles to the axis AB, in circles whose radii are EF, EG, EH.

Then the areas of these circles are proportional to EF^2, EG^2, EH^2 respectively.

Therefore the volumes of the segments of these solids included between the parallel planes are also ultimately proportional to EF^2, EG^2, EH^2.

But since $$EF = EA;$$

$$\therefore\ EF^2 + EG^2 = EA^2 + EG^2 = AG^2 = AC^2 = EH^2;$$

and therefore the slices of the cone and hemisphere together equal the slice of the cylinder.

And therefore, taking the sums of all the slices,

vol. of cone + vol. of hemisphere = vol. of cylinder.

But vol. of cone $= \frac{1}{3}$ vol. of cylinder (Th. 33, Cor. 1);

$\therefore$ vol. of hemisphere $= \frac{2}{3}$ vol. of cylinder;

and $\therefore$ vol. of sphere $= \frac{2}{3}$ vol. of circumscribing cylinder[1].

COR. *If* r *is the radius of the sphere, its volume is thus shewn to be*

$$\frac{2}{3} \times 2r \times \pi r^2$$

$$= \frac{4}{3}\pi r^3.$$

[1] It was Archimedes who discovered that the volume and surface of the sphere are each $\frac{2}{3}$rds of that of the circumscribing cylinder. He directed that the figures by which this result was obtained should be carved on his tomb.

NOTE. This result might be deduced from Theorem 29, by regarding the sphere as the limit of a polyhedron, the number of whose faces was indefinitely increased. For if pyramids were formed having as their bases the faces of the polyhedron, and common vertex the centre of the sphere;

each pyramid $= \frac{1}{3}$ height × face.

And therefore sum of pyramids $= \frac{1}{3}$ height × sum of faces.

But ultimately the height of the pyramid = radius r, and sum of faces of polyhedron = surface of sphere

$$= 4\pi r^2; \qquad \text{(Th. 39.)}$$

$$\therefore \text{ volume of sphere} = \frac{1}{3} r \times 4\pi r^2$$

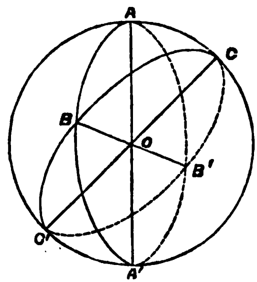
$$= \frac{4}{3}\pi r^3.$$

SPHERICAL TRIANGLES.

From our referring the apparent position of the heavenly bodies to an imaginary spherical surface, of which we occupy the centre, the geometry of the surface of the sphere early attracted attention. It cannot be studied to any great extent without a knowledge of Trigonometry, but a few propositions may be given which are of importance, and illustrate this branch of Geometry.

Since the planes of two great circles both pass through the centre of the sphere, they must intersect in a diameter of the sphere.

Therefore in the figure, if ABA', CBC', ACA' are great circles, AA', BB', CC' are diameters; and A is then said to be antipodal to A'.



Further since AA', BB' intersect in the centre O of the sphere, the angles AOB, $A'OB'$ are equal; and therefore also the arcs AB, $A'B'$ are equal.

Def. 30. *A spherical triangle* is the portion of the surface of a sphere bounded by three great circles.

Thus in the figure ABC and $A'BC'$ and ABC' are spherical triangles.

The triangle ABC is said to be *symmetric* with $A'B'C'$ when the vertices of the one are respectively antipodal to the vertices of the other.

In a spherical triangle ABC we have to consider three arcs AB, BC, CA, which are called its *sides*, and three angles, A, B, and C.

If A, B, C are joined to O the centre of the sphere, then the sides AB, BC, CA are proportional to the angles AOB, BOC, COA, of the trihedral angle at O.

Further the angle ABC of the spherical triangle is the angle between the tangents at B to the circles AB, BC

respectively; and since these tangents are at right angles to OB, and in the planes of the circles AB, BC; therefore the angle between the tangents measures the dihedral angle between the planes of the circles; and therefore the angles of the spherical triangle ABC are equal to the dihedral angles between the faces of the trihedral angle $OABC$.

We may therefore speak of *all* the parts of a spherical triangle as *angles*, meaning thereby the angles of the faces, and the dihedral angles between the faces, of the trihedral angle whose vertex is the centre of the sphere, and base the spherical triangle.

The properties therefore of a spherical triangle and a trihedral angle are mutually convertible.

The portion of the surface of the sphere included between *two* great circles, as $AC'A'B$, is called a spherical segment; and its area has to the surface of the whole sphere the same ratio that the angle between the planes of the circles has to four right angles.

The area therefore of a spherical segment is measured by its angle.

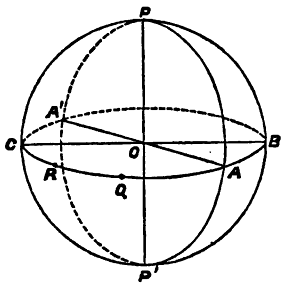
The *poles* of a great circle are the extremities of the diameter at right angles to its plane, and the great circle is called the polar circle of that point.

Thus if POP' is at right angles to the plane of the great circle $AOBC$, and meets the surface in P, P', then P, P' are the poles of the circle ABC.

The arcs AP, BP, subtending right angles at the centre O, are called *quadrantal arcs*.

The angle between the planes, or the angle at P of the spherical triangle APB, is equal to AOB, and is therefore measured by the arc AB.

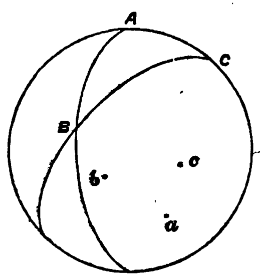
Since the poles of *BPC*, *APA′* must be at the distance of a quadrant from *P*, they must lie on *BAC*.



Hence the polar circle of any point *P* contains the poles of all circles which intersect in *P*.

If *Q*, *R* are the poles respectively of the circles *PBP′*, *PAP′*; then since *RA* and *QB* are quadrants, therefore *RQ* = *AB*.

Therefore *RQ* measures the angle at *P*; that is, the arc between the poles of two great circles measures the angle at which they intersect.



Therefore if *a*, *b*, *c* are respectively the poles of the circles *CB*, *AC*, *BA*, the *sides* of the triangle *abc* are the measures of the *angles* of the triangle *ABC*.

abc is then called the ***polar triangle*** of ABC.

Moreover, since b is the pole of AC, and c is the pole of AB, the point A is distant by one quadrant from both b and c; that is, A is the pole of bc.

Similarly B, C are the poles of ca, ab respectively.

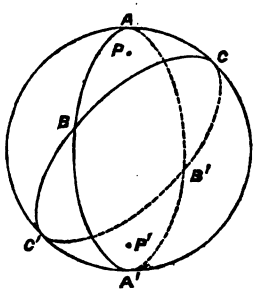
Hence if abc is the polar triangle of ABC, then ABC is the polar triangle of abc.

THEOREM 41.

A spherical triangle is equal to its symmetric spherical triangle.

Let ABC be a spherical triangle, A', B', C', the antipodal points to A, B, C.

Then will the triangle ABC be equal to the triangle $A'B'C'$.



(1) Let the triangles be isosceles, i.e. let $AB = BC$, and $A'B' = B'C'$.

If the angle B were placed on B', the concavities of the sphere being on the same side, the side BA would fall on $B'C'$, and BC on $B'A'$,

and since $BA = B'C'$ and $BC = B'A'$;

therefore A would fall on C' and C on A', and the triangles would wholly coincide and be equal.

(2) Let the triangles not be isosceles.

Let a perpendicular be let fall from the centre on the plane ABC, meeting the surface in P, then since the perpendicular from the centre of the sphere on a small circle of the sphere passes through its centre, P will be the pole of the circle ABC, and therefore the arcs AP, BP, CP are all equal. Similarly if P' is the antipodal point to P, the arcs $P'A'$, $P'B'$, $P'C'$ are all equal to PA, PB, PC.

Hence each of the triangles PAC, PCB, PBA is isosceles, and equal respectively to their symmetric triangles $P'A'C'$, $P'A'B'$, $P'B'C'$, and therefore the whole triangle ABC is equal to the whole triangle $A'B'C'$.[1]

THEOREM 42.

The sum of the three angles of a spherical triangle none of whose sides or angles exceeds two right angles is always greater than two right angles.

Let ABC be a spherical triangle (*vide* fig. of last Theorem); then since the sum of the three spherical segments $ABA'C$, $A'BC'B'$, $ACBC'$ exceeds the hemisphere ACA' by the two triangles ABC, $A'B'C'$,

but the measures of the three spherical segments are respectively the angles A, B, C of the spherical triangle;

and the measure of the hemisphere is two right angles;

[1] This proof was first published by Legendre; the discoverer of it is not known.

therefore the sum of the three angles exceeds two right angles.

COR. *The excess of the sum of the three angles of a triangle over two right angles is a measure of its area.*

If A is the number of degrees in the angle A, S the surface of the hemisphere, the area of the spherical segment $= \frac{A}{180} \cdot S$; therefore, since ABC is equal to its symmetric triangle $A'B'C'$, the result of the theorem is that if Σ is the area of the spherical triangle, S the surface of the hemisphere,

$$\left(\frac{A+B+C}{180} - 1\right) S = 2\Sigma,$$

$$\text{or } \Sigma = \frac{A+B+C-180}{360} \cdot S,$$

that is, the area Σ is proportional to the excess of $A+B+C$ over two right angles.

Hence spherical triangles on the same sphere are equal when the sums of their angles are equal.

EXERCISES ON SECTION IV.

1. Find the surface of a sphere 25 inches in diameter ($\pi = 3\frac{1}{7}$).

2. Find the radius of a sphere that shall contain exactly a cubic yard.

3. Find the weight of a 10 inch shell of iron, the iron being 1 inch thick, and weighing 444 lbs. to the cubic foot.

4. Prove that a sphere is inscribable in a cylinder, and will touch it along the circumference of a circle.

5. To draw a tangent plane to a sphere so that the plane shall contain a given line.

6. To draw a plane through a given point within a sphere so that the area intercepted by the sphere on the plane shall be a minimum.

7. Prove that small circles on the sphere of equal radii have their planes equally distant from the centre.

8. Given two points on a given sphere, to describe the great circle passing through them.

9. Prove that the perpendiculars to the three faces of a tetrahedron through the centres of the circles circumscribed to these faces will all intersect in one point.

10. Bisect a given arc of a great circle.

11. Given three points on a sphere, to describe a small circle to pass through them.

12. If three spheres intersect one another, their planes of intersection intersect in a line perpendicular to the plane containing the centres of the spheres.

Prove that this line is the locus of points from which tangent lines to the three spheres are equal.

MISCELLANEOUS EXERCISES IN GEOMETRY OF SPACE.

1. If a straight line is at right angles to three straight lines which intersect it at one point, these three lines will be in one plane.

2. To find a point in a given straight line equally distant from two points in space.

3. To find the locus of points in space such that the difference of the squares of their distances from two given points is constant.

4. To draw a straight line of given length, and parallel to a given plane, with its extremities on two straight lines in space, the straight line of given length being longer than the perpendicular distance between the given straight lines.

5. Find the locus of points equally distant from the edges of a trihedral angle.

6. Find the locus of points equally distant from the faces of a trihedral angle.

7. If a trirectangular trihedral angle be cut by a plane at distances a, b, c from the vertex, prove that the square of the area of the section $= \frac{1}{4}\{a^2b^2 + a^2c^2 + b^2c^2\}$.

8. To draw a plane parallel to two given straight lines in space.

9. Prove that the sum of the squares of the four diagonals of a parallelepiped is equal to the sum of the squares of its edges.

10. To find a point within a tetrahedron such that by joining it to the angular points the four tetrahedra so determined are equivalent.

11. Express the surface, altitude and volume of a regular tetrahedron whose edge $= a$.

12. Solve the same problem in the case of the octahedron.

13. To cut a cube by a plane so that its section shall be a regular hexagon.

14. Given three lines, no two of which are in the same plane, to construct a parallelepiped three of whose edges are in these lines.

15. If a prism or a pyramid be cut by a plane not parallel to the base and the corresponding sides of the sections be produced till they meet, prove that the points of intersection will all lie in one straight line.

Deduce from this by projections a theorem in plane Geometry.

16. Four planes intersect one another so as to form a tetrahedron; shew that eight spheres can in general be described so as to touch them all.

17. If two pairs of opposite edges of a tetrahedron are mutually at right angles, the third pair will also be at right angles.

18. If the three pairs of opposite edges of a tetrahedron are mutually at right angles, prove that the four altitudes of the tetrahedron pass through one point.

19. Prove that the six planes which bisect the edges of a tetrahedron at right angles will all pass through one point.

20. A boiler is a cylinder with hemispherical ends. If the total length is 20 feet and circumference 11 feet, find its surface and the quantity of water required to fill it half full.

21. Find the volume of the double cone generated by the revolution of an equilateral triangle about one of its sides.

22. AB is an arc of sphere, prove that the surface of the sphere bounded by a circle whose centre is A and radius AB will $= \pi . AB^2$.

23. A cone is circumscribed to a sphere, and its height is double the diameter of the sphere. Prove that the total surface and the volume of the cone are respectively double of those of the sphere.

24. A rectangle revolves in succession round two of its unequal sides, prove that the volumes of the cylinders generated are inversely proportional to the lengths of the sides round which it revolves.

25. Shew that a regular dodecahedron may be inscribed in a regular icosahedron; and conversely, that a regular icosahedron may be inscribed in a regular dodecahedron.

Is there any other pair of regular polyhedra so related to each other?

26. To describe a sphere to cut orthogonally two given spheres.

27. To bisect a given arc, or given angle, on a sphere.

28. Three lines mutually at right angles intersect a sphere; prove that the sum of the squares of the three chords is constant, depending only on the radius of the sphere and the distance of the point from the centre.

Prove also that the sum of the squares of the six segments is constant.

29. If through a point O any secant OPP' is drawn to cut a sphere in P, P', prove that $OP \cdot OP'$ is constant.

30. Find the radii of the spheres inscribed and circumscribed to a regular tetrahedron.

APPENDIX I.

TRANSVERSALS.

Positive and Negative Signs in Geometry.

If a point P is conceived to traverse a line in which there are two fixed points A, B, the lines AP, BP are called the *segments* of the line AB, made by the point P,

whether P divides AB internally or externally. And AP and BP are considered to have the same signs when they are measured in the same sense, and to have opposite signs when they are measured in the opposite sense.

Thus AP and BP have the same sign, AP' and BP' have opposite signs, AP'', BP'' have the same sign.

It is usual to consider AP'', BP'', and lines measured in this sense as positive; and AP, BP' as negative.

Hence, if P divides AB internally, the ratio $\frac{AP}{BP}$ is negative, and if externally $\frac{AP}{BP}$ is positive.

The result of this is very important. For if P be supposed to traverse the indefinite line through AB from left to right, the ratio $\frac{AP}{BP}$ at first is +, and as P moves to A passes through all values from $+1$ to 0; as P passes through A, $\frac{AP}{BP}$ passes through zero and becomes negative; and as P moves from A to B, $\frac{AP}{BP}$ passes through all values from 0 to $-\infty$; and as P passes through B towards the right $\frac{AP}{BP}$ changes from $-\infty$ to $+\infty$, is again +, and passes through all values from ∞ to 1.

Hence taking into account the sign as well as the magnitude of the ratio $\frac{AP}{BP}$, it appears that *there is one position, and only one, of the point* P *which makes the ratio* $\frac{\mathrm{AP}}{\mathrm{BP}}$ *equal to a given ratio.*

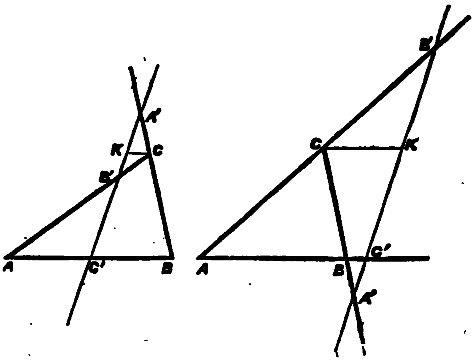
Def. A *transversal* to a triangle is any straight line drawn to intersect its three sides or sides produced.

THEOREM I.

A transversal determines six segments on the sides of a triangle, such that the products of the alternate segments are equal, the signs of the segments as well as their magnitudes being considered.

Let ABC be a triangle, and let the transversal $A'B'C'$ intersect the sides opposite to A, B, C in A', B', C'.

Then will the product of the segments AC', BA', CB' be equal to the product of the segments BC', CA', AB'.



Through C draw CK parallel to AB to meet the transversal in K.

Then by similar triangles $A'CK$, $A'BC'$,

$$\frac{CA'}{BA'} = \frac{CK}{BC'}.$$

And by similar triangles CKB', $AC'B'$,

$$\frac{AB'}{CB'} = \frac{AC'}{CK},$$

$\therefore$ multiplying these ratios,

$$\frac{CA' \,.\, AB'}{BA' \,.\, CB'} = \frac{AC'}{BC'},$$

or $BC' \,.\, CA \,.\, AB' = AC' \,.\, BA' \,.\, CB'$.

Conversely. If points A′, B′, C′ *so divide the sides of a triangle, that the products of the alternate segments are equal, then* A′, B′, C′ *are collinear.*

This follows at once by the *reductio ad absurdum.*

For if $A'B'$ intersected AB not in C', but in some other point C'', then by the theorem given above the product of the ratios $\frac{AC''}{BC''}, \frac{BA'}{CA'}, \frac{CB'}{AB'}$ would equal unity. But by hypothesis the product of the ratios

$$\frac{AC'}{BC'}, \frac{BA'}{CA'}, \frac{CB'}{AB'} = \text{unity},$$

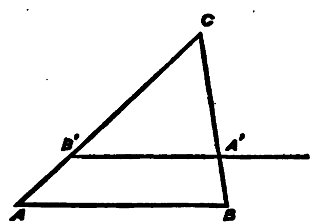
and $\therefore \frac{AC''}{BC''} = \frac{AC'}{BC'}$, which is impossible.

Some special cases of the Theorem may now be examined.

COR. I. If the transversal is parallel to AB, C' is at an infinite distance, and $\frac{AC'}{BC'} = 1$.

Therefore the formula reduces itself to

$$\frac{CA' \cdot AB'}{BA' \cdot CB'} = 1,$$



or the well-known case

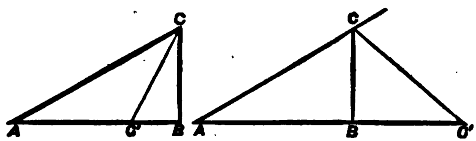
$$BA' : CA' :: AB' : CB'.$$

COR. 2. If $CA' = CB'$ in absolute magnitude, and sign, in fig. 1, or fig. 2, then $CA'B'$ is an isosceles triangle; and the transversal is parallel to the internal bisector of the angle C,

$$\therefore \frac{BC'.AB'}{AC'.BA'} = \frac{CB'}{CA'} = 1.$$

Further, if the transversal is drawn through C, so that A' and B' coincide with C, this reduces to the well-known Theorem

$$BC' : AC' :: BC : AC.$$



Similarly, by making $CA' = -CB'$, we may deduce that when the transversal bisects the exterior angle

$$BC' : AC' :: BC : CA.$$

It will be noticed that the transversal will cut either two sides internally, or none internally; and therefore that of the three ratios $\frac{AC'}{BC'}, \frac{BA'}{CA'}, \frac{CB'}{AB'}$, either two will be negative, or none, and hence the product of the three will always be positive.

Def. Lines are said to be *concurrent* when they pass through one point.

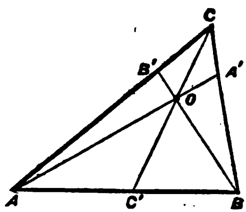
THEOREM 2.

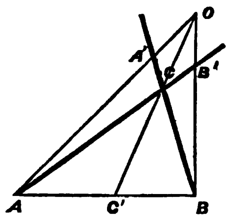
If three concurrent lines from the vertices A, B, C *of a triangle intersect the opposite sides in* A′, B′, C′, *then*

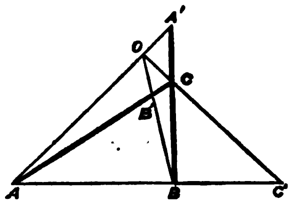
$$\frac{AC'\,.\,BA'\,.\,CB'}{BC'\,.\,CA'\,.\,AB'} = -1.$$

Since AC', BC' are proportional to the altitudes of the two triangles AOC, BOC, which have a common base OC,

$$\therefore \frac{AC'}{BC'} = \frac{AOC}{BOC}.$$







Similarly

$$\frac{BA'}{CA'} = \frac{BOA}{COA} \text{ and } \frac{CB'}{AB'} = \frac{COB}{AOB},$$

therefore, multiplying these ratios, and observing that of the three ratios either one or three must be negative, we obtain

$$\frac{AC' \cdot BA' \cdot CB'}{BC' \cdot CA' \cdot AB'} = -1.$$

Conversely, if $\dfrac{AC' \cdot BA' \cdot CB'}{BC' \cdot CA' \cdot AB'} = -1$,

then AA', BB', CC' are concurrent.

This may be proved by *reductio ad absurdum*.

Example.

If the internal and external bisectors of the angles of a triangle ABC meet the opposite sides in A', B', C', A'', B'', C'', respectively, these points lie three and three in four straight lines.

To prove this we must establish the condition of collinearity given in Theorem 1.

Thus we have

$$\frac{AC'}{BC'} = -\frac{CA}{BC},$$

$$\text{and } \frac{BA'}{CA'} = -\frac{AB}{CA}, \text{ \&c.}$$

$$\text{and } \frac{CB''}{AB''} = \frac{BC}{AB};$$

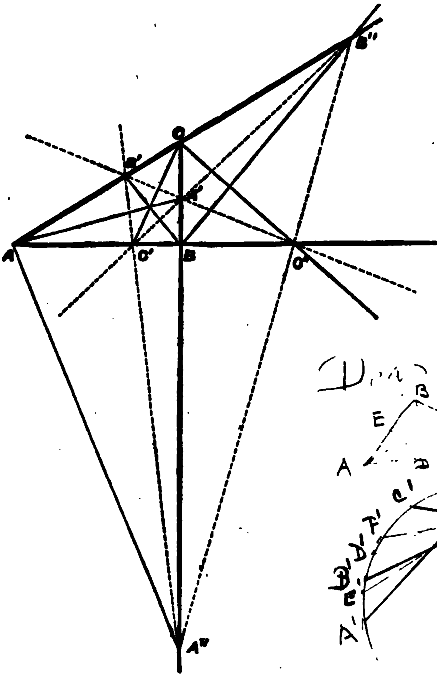
$$\therefore \frac{AC' \cdot BA' \cdot CB''}{BC' \cdot CA' \cdot AB''} = 1,$$

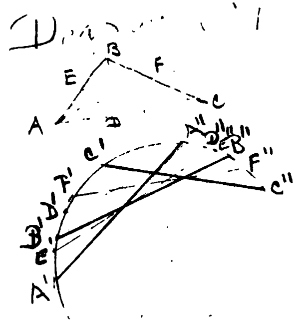
and C', A', B'' are collinear by Th. 1.

Similarly B', A', C'' are collinear,

and B', C', A'' are collinear;

and in the same way it may be shewn that A'', B'', C'' are collinear.





EXERCISES ON TRANSVERSALS, &c.

1. Prove that the three bisectors of the sides of a triangle, drawn from the opposite vertices, intersect in one point.

2. Prove that the three bisectors of the angles of a triangle are concurrent.

3. Prove that the lines joining the vertices of a triangle to the points of contact of the inscribed circle are concurrent.

4. Prove that the three perpendiculars of a triangle are concurrent.

5. When three of the six intersections of a circle with the sides of a triangle connect concurrently with the opposite vertices, prove that the other three have the same property.

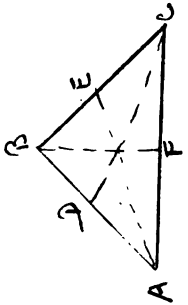
6. If A, B, C, D are *any* four points on a straight line, prove that $AB + BC + CD + DA = 0$.

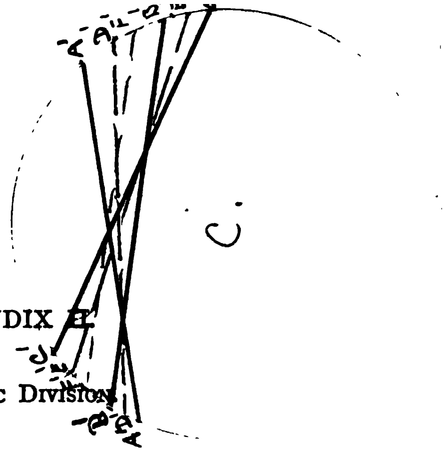
7. If ABC is a triangle inscribed in a circle, and the tangent at A meets BC produced in a, prove that

$$\frac{Ca}{Ba} = \frac{\overline{CA}^2}{\overline{BA}^2}.$$

Hence prove that the points of intersection of the sides of the inscribed triangle with the tangents at the vertices are collinear.

8. When three lines through the vertices of a triangle are concurrent, their three points of intersection with the opposite sides determine an inscribed triangle whose sides intersect collinearly with those of the original triangle to which they correspond.





APPENDIX II.

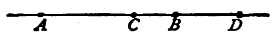
Harmonic Division.

Def. Three quantities a, b, c are said to be *in Harmonic progression* when $\frac{a-b}{b-c}=\frac{a}{c}$.

This relation is easily seen to be identical with either of the following relations:

$$\frac{1}{a}+\frac{1}{c}=\frac{2}{b}, \text{ or } b=\frac{2ac}{a+c}.$$

Def. A line AB is said to be *harmonically divided* in C, D,



when if $AD=a$, $AB=b$, $AC=c$, a, b, c are in harmonic progression.

This by the definition above leads to the relations

$$\frac{BD}{CB}=\frac{AD}{AC} \text{ or } \frac{BD}{AD}=-\frac{BC}{AC},$$

$$\text{or } \frac{DA}{CA}=-\frac{BD}{BC},$$

$$\text{or } AC\times BD=CB\times AD,$$

that is, the product of the exterior segments = the whole line × middle segment;

$$\therefore \text{ also } AC : CB :: AD : DB.$$

Hence if AB is harmonically divided by C, D, CD is also harmonically divided by A, B.

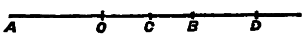
The points A, C, B, D are said to form a *harmonic range.*

A special case of this is when C bisects AB: in this case since $\frac{AC}{CB}=\frac{AD}{BD}$, it follows that $\frac{AD}{BD}=1$; that is, D is at an infinite distance.

The points A, B are said to be *conjugate* to one another in the harmonic range $ACBD$, and likewise the points C, D are conjugate to one another.

THEOREM 1.

If ACBD *form a harmonic range, and* AB *is bisected in* O, *then* OC . OD = OB².



Since $AC \times BD = AD \times CB$,

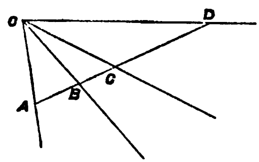
$$\therefore (AO+OC)(OD-OB)=(AO+OD)(OB-OC),$$

and multiplying out, and observing that $AO=OB$, it follows that $OC \times OD = OB^2$.

The converse proposition is also true.

Def. Any number of straight lines meeting in one point are called a pencil; and each of the lines is called a ***ray.***

The pencil is *harmonic* when any transversal is harmonically divided.

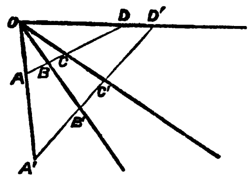


The figure is described as the pencil $O\{ABCD\}$, or $O.ABCD$.

THEOREM 2.

If a pencil divides any transversal harmonically, it will divide all transversals harmonically.

Let the pencil $O.ABCD$ divide the transversal $ABCD$ harmonically, so that



$$AB.CD = BC.AD.$$

Then will it divide any other transversal $A'B'C'D'$ harmonically.

For since two triangles, which have an angle equal, have to each other the ratio compounded of the ratio of their sides,

$$\therefore \triangle AOB : \triangle A'OB' :: AO.OB : A'O.OB',$$

$$\text{and } \triangle COD : \triangle C'OD' :: CO.OD : C'O.OD';$$

$$\therefore \frac{AOB \times COD}{A'OB' \times C'OD'} = \frac{AO.BO.CO.DO}{A'O.B'O.C'O.D'O};$$

$$\text{similarly } \frac{BOC \times AOD}{B'OC' \times A'OD'} = \frac{AO.BO.CO.DO}{A'O.B'O.C'O.D'O};$$

$$\therefore \frac{AOB \times COD}{A'OB'.C'OD'} = \frac{BOC \times AOD}{B'OC' \times A'OD'};$$

therefore, observing that triangles of equal altitude are to one another as their bases,

$$\frac{AB \cdot CD}{BC \cdot AD} = \frac{A'B' \cdot C'D'}{B'C' \cdot A'D'};$$

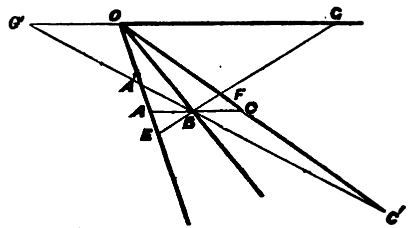
but $AB \cdot CD = BC \cdot AD$, by hypothesis;

$$\therefore A'B' \cdot C'D' = B'C' \cdot A'D',$$

which proves the theorem.

Cor. 1. *If a transversal parallel to one ray is bisected by its conjugate, then the pencil is harmonic.*

This is a special case of the theorem, or may easily be proved independently, as follows.



Draw a transversal through B meeting the rays in $EBFG$.

Then by similar triangles $\frac{EB}{EG} = \frac{AB}{OG}$ and likewise

$$\frac{FG}{BF} = \frac{OG}{BC},$$

therefore multiplying,

$$\frac{EB \times FG}{EG \times BF} = \frac{AB}{BC} = 1.$$

Similarly, if the transversal meets GO produced backwards through O, the pencil $O\{G'A'BC'\}$ is harmonic.

Def. The line OG is called the *polar* of B with reference to the angle AOC; and B is called the pole of OG.

COR. 2. *It follows from what has been said that if the line* EBFG *rotate round* B, *and the point* G *is always taken on its conjugate to* B, *with reference to* E, F, *the locus of* G *is the straight line* OG *passing through the vertex* O *of the angle* AOC.

Hence we obtain the following definition:

Def. The polar of O with reference to an angle is the locus of points conjugate to O with reference to the extremities of transversals to the angle drawn through O.

COR. 3. *If* OB *bisects the* $\angle$ AOC, AC *and* $\therefore$ OG *is at right angles to* OB.

HARMONIC PROPERTIES OF THE COMPLETE QUADRILATERAL.

Def. A complete quadrilateral is formed by taking any four points, joining each pair, and the points of intersection of each pair.

THEOREM 3.

In a complete quadrilateral all the pencils are harmonic pencils.

Let $ABCD$ be a quadrilateral, which may be either convex as in the figure, or have one of its points within the triangle formed by the other three.

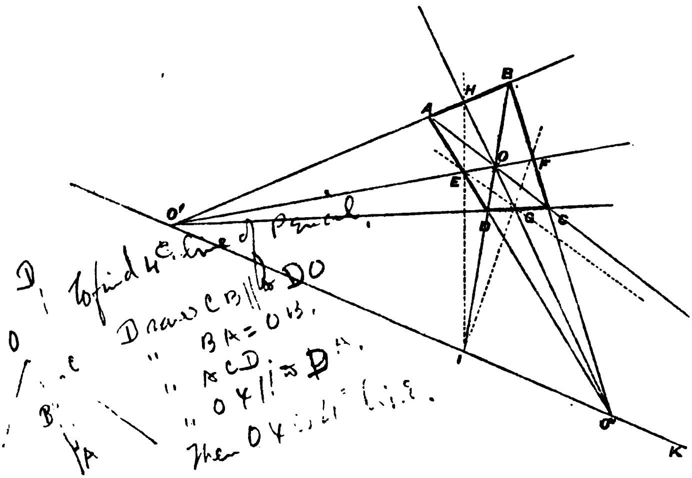
Let AC, BD intersect in O,
... BA, CD .... O',
... AD, BC .... O''.

Join OO', OO'', $O'O''$.

Then AC, BD, $O'O''$ are called its three diagonals.

All the pencils in the figure will be harmonic.

First, to shew that the pencil at O' is a harmonic pencil.



Let E, F be the conjugate points to O'' on the lines AD, BC so that $AEDO''$, $BFCO''$ are harmonic ranges.

Then considering the angle $AO'D$, and the pole O'', it follows from Th. 2, Cor. 2, that E and F both lie on the polar of O'' with respect to that angle, and therefore EF passes through O'.

Similarly considering the angle AOD or BOC, and the pole O'', we infer that EF passes through O.

Therefore $O'O$ is the polar of O'' with reference to the angle $AO'D$, or the pencil at O' is harmonic.

Secondly. All the lines in the figure are divided harmonically.

If OO'' meet DC, AB in G, H, in the same way it may be shewn that O'' is a harmonic pencil, and therefore the pencils that meet in O, O', O'' are all harmonic, and every line in the figure is a transversal of one of these pencils, and therefore harmonically divided.

It must be noticed that of the lines OO', $O'O''$, $O''O$, each is the polar of the intersection of the other two with reference to the angle formed by the two sides of the quadrilateral which intersect on it.

Thirdly, if AC, $O'O''$ intersect in K, then E, G, K, and I, G, F, and I, E, H, lie respectively in a straight line.

Since $AEDO''$ forms a harmonic range, the pencil $K(AEDO'')$ is a harmonic pencil.

Similarly $K(CGDO')$ is a harmonic pencil; but three of the rays of these pencils are identical; therefore the fourth is identical, that is, KGE is a straight line.

Similarly since $I(BFCO'')$ is a harmonic pencil,

and also $I(O'DGC)$ ....

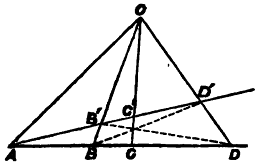
and three rays are identical, therefore the fourth is identical, therefore IGF is a straight line.

In the same manner I, E, H are collinear.

THEOREM 4.

If ABCD, AB′C′D′ *are harmonic ranges,* BB′, CC′, DD′ *will intersect in one point, and likewise* BD′, CC′, B′D *intersect in one point.*

For let BB', CC' intersect in O, and join AO, then the fourth ray of the harmonic pencil at O must pass through D and D', and $\therefore$ DD' passes through O.



Also DB', BD' intersect on OC, for the polar of A with reference to the angle BOD is (by Th. 3) the line joining the intersection of BB', DD' with that of $B'D$, BD'.

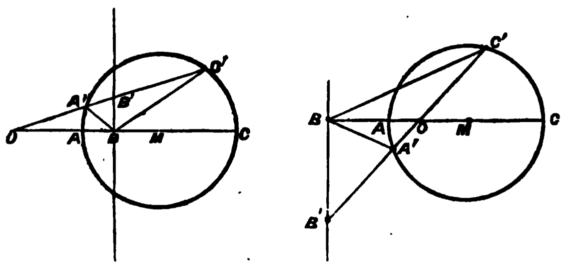
(Pole of

HARMONIC PROPERTIES OF THE CIRCLE.

Def. If through a fixed point O a chord be drawn to meet a fixed circle in AC, and B be the conjugate to O with reference to A, C, then the locus of B is called the polar of O, and O itself is called the pole.

THEOREM 5.

The polar of a point with reference to a circle is a straight line.



Let O be a point without or within a circle, $OABC$ the diameter through O, M the centre, B the point conjugate to O.

Then B is determined by the condition $MB \cdot MO = MA^2$ (Th. 1).

Let $OA'C'$ be any other chord, B' the conjugate point to O on that chord;

Then since $OA : AB :: OC : BC$, therefore the circle on AC as diameter is, by Bk. III. Th. 5, the locus of points whose distances from O, B are in a constant ratio,

$$\therefore \frac{OA'}{A'B} = \frac{OC'}{C'B},$$

and therefore
$$\frac{A'B}{C'B} = \frac{OA'}{OC'}.$$

And therefore OB bisects the exterior angle of the triangle $A'BC'$, that is, one of the angles between the rays, and therefore (Th. 2, Cor. 3) OBB' is a right angle;

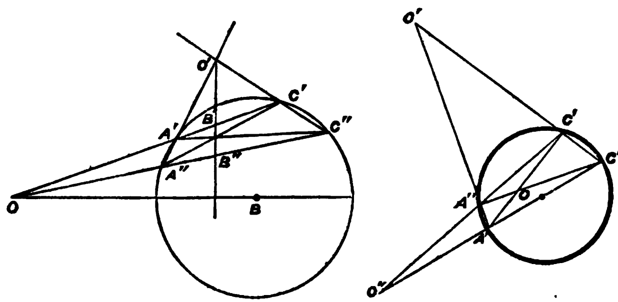
$\therefore$ the locus of B is the perpendicular to OC through B determined by the condition $MO \, . \, MB = MA^2$.

Theorem 6.

The polar of a point with reference to a circle is the locus of the intersections of the lines which join the extremities of chords passing through that point.

Let $OA'C'$, $OA''C''$ be two chords; B', B'' the conjugate points to O, then $B'B''$ is the polar of O. (Th. 5.)

Join $A'A''$, $C'C''$. Then since $OA'B'C'$, $OA''B''C''$ are harmonic ranges with the point O common, $A'A''$, $B'B''$,



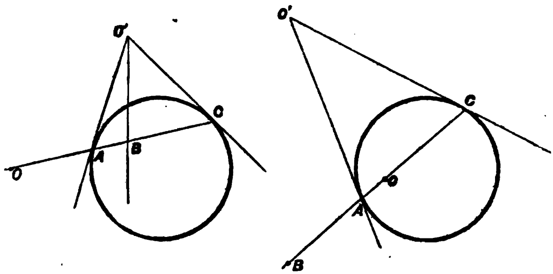
$C'C''$ intersect in one point, i.e. the lines $A'A''$, $C'C''$ intersect on the polar of O. (Th. 4.)

Similarly (by Th. 4) $A'C''$ and $A''C'$ intersect on $B'B''$.

COR. 1. A special case of this is when the chords $OA'C'$, $OA''C''$ coincide; for then the chords $A'A''$, $C'C''$ become the tangents at A', C';

Hence *the locus of the intersection of tangents at the extremities of chords through any point is the polar of that point.*

COR. 2. *If the point* O *is on the circumference of the circle, its polar coincides with the tangent at that point.*



COR. 3. *Since the polar of* O *passes through* O′, *and the polar of* O′ *through* O, *we see that the polar of every point on a straight line passes through the pole of that line.*

COR. 4. *In the triangle* OBO′ *each side is the polar of the opposite angle.*

EXERCISES.

1. The bisectors of the internal and external angles of a triangle form with the sides a harmonic pencil.

2. Given three rays of a harmonic pencil, to find the fourth.

3. Given three points of a harmonic ray, to find the conjugate to one of them with respect to the other two.

4. If two of the conjugate rays of a pencil are at right angles to one another, prove that they bisect the angles between the other two.

5. Prove that the polars of any point, with reference to the angles of a triangle, intersect the opposite sides in three collinear points.

6. If $ABCD$ be any four points on a line, prove that

$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC = 0.$$

7. If a quadrilateral figure be described about a circle, and the points of contact of opposite sides be joined, prove that these lines and the diagonals of the quadrilateral figure all intersect in one point; that they pass through the intersections of opposite sides of the figure; and the lines which join the points of contact of adjacent sides intersect in pairs on the straight lines that join the intersections of opposite sides of the figure.

8. If $ABXY$ are four points on a circle, which determine a harmonic pencil at any point P on the circle, then they determine at every other point on the circle a harmonic pencil.

CONIC SECTIONS.

BOOK V.

CHAPTER I.

ON THE SECTIONS OF A CONE.

A *right circular cone* is the solid generated by the revolution of a right-angled triangle round one of the sides containing the right angle.

The fixed side is called the *axis* of the cone.

The hypothenuse, which by its motion generates the surface of the solid, is in any position called a *generating line*, which meets the axis in a point called the *vertex*.

THE PARABOLA.

Def. The section of a cone made by a plane which is parallel to one of the generating lines of the cone, and perpendicular to the plane which contains that generating line and the axis of the cone, is called a *parabola*.

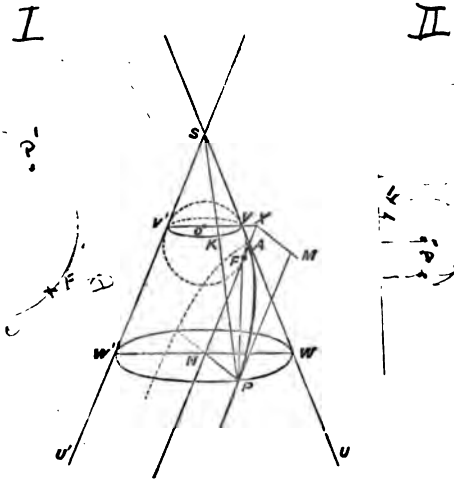
THEOREM I.

In the parabola the distance of every point on the curve from a fixed point in its plane is equal to its distance from a fixed straight line, also in its plane.

Let the plane of the paper contain the axis of a right circular cone, and intersect its surface in the generating lines SU, SU'; and let a plane, perpendicular to the

plane of the paper, and parallel to SU', intersect the cone in the parabola AP, and the plane of the paper in the line AN.

Let a sphere be described to touch the cone in the circle VKV', and the plane of the parabola in the point F, its centre being in the plane of the paper. (Th. 36.)



Let the plane of circle VKV' intersect the plane of the parabola in the line XM, which will be perpendicular to the plane of the paper. IV. 18, Cor.

Take any point P on the parabola. Join SP, meeting the circle VKV' in K; join FP, and draw PM perpendicular to XM.

Draw a plane through P perpendicular to the axis of the cone, to cut the cone in the circle WPW', and the plane of the parabola in PN, which will also be perpendicular to the plane of the paper.

Then $FP = KP$, being tangents from P to a sphere.

But since $SP = SW$, and $SK = SV$,

$$\therefore\ KP = VW;$$

and since AN is parallel to SW', by the definition of a parabola, $\therefore$ the angle $ANW = SW'W$

$$= SWW',$$

and therefore ANW is isosceles.

So also AVX is isosceles; and therefore $VW = XN$.

But $XN = PM$, being opposite sides of a parallelogram, and therefore $FP = PM$:

that is, the distance of P, any point on a parabola, from a fixed point F in its plane is equal to its distance from a fixed line XM, also in its plane.

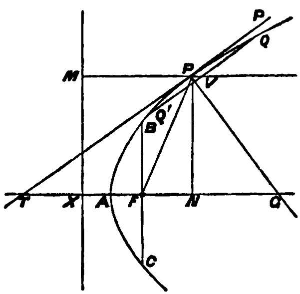
COR. *The parabola is symmetrical with respect to the axis* AN.

DEFINITIONS.

The following are definitions of terms used in studying Conic Sections.

The fixed point F is called *the focus*.

The line XM is called *the directrix*.



If FX, perpendicular to XM, meet the curve in A, A is called the *vertex*, and AF produced is called *the axis*.

A straight line PN perpendicular to the axis is called the *ordinate* of P; AN is its *abscissa*.

The double ordinate through the focus is called the *Latus Rectum*.

A line drawn to cut the curve is called a *secant*.

A line drawn to touch the curve at P is called *the tangent* at P; PG perpendicular to the tangent at P, and meeting the axis in G, is called the *normal*.

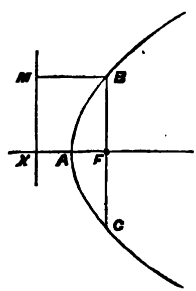
NT is called the *subtangent*; NG the *subnormal*.

A line MPV parallel to the axis of a parabola is called a *diameter*, and a line QV parallel to the tangent at P is called an *ordinate to the diameter through P*; PV is the corresponding *abscissa*. The focal chord parallel to PT is called the *parameter* of the diameter through P.

Theorem 2. The Latus Rectum.

The Latus Rectum BC = 4AF.

Let BC be the latus rectum; draw BM perpendicular to the directrix.



Then $BF = BM$, by the property of the parabola,

$$= XF$$

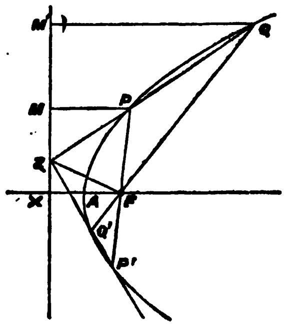
$$= 2AF,$$

since $AF = AX$, by the property of the parabola;

$$\therefore BC = 4AF.$$

THEOREM 3. THE SECANT.

If a secant PQ *meets the directrix in* Z, ZF *is the bisector of the exterior angle between the focal distances* FP, FQ.



Draw PM, QM' perpendicular to the directrix:

Then, by similar triangles ZPM, ZQM',

$$PZ : QZ :: PM : QM'$$
$$:: PF : QF,$$

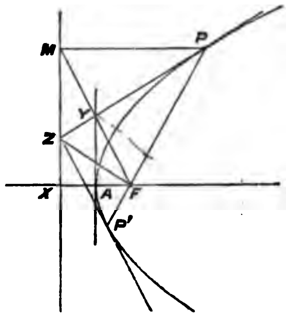
$\therefore$ FZ is the exterior bisector of PFQ. Eucl. VI. A.

COR. 1. *If* PF, QF *produced meet the curve again in* P′, Q′, FZ *is also the bisector of the exterior angle between* P′F, Q′F; *therefore* P′Q′ *passes through* Z.

COR. 2. PQ′, QP′ *produced intersect on the directrix in some point* Z′, *such that* FZ′ *bisects the angle* Q′FP′ *by Cor.* 1; *and therefore* FZ, FZ′ *are the bisectors of the adjacent angles* PFQ′, Q′FP′; *and therefore* ZFZ′ *is a right angle.*

THEOREM 4. THE TANGENT.

The tangent at P *bisects the angle between the focal distance of* P *and the perpendicular from* P *on the directrix, and* PZ *subtends a right angle at the focus.*



The tangent at P is the limiting position of the secant PQ in the figure of Theorem 3, when Q moves up to P: and therefore FQ coincides with FP.

Therefore if PZ is the tangent at P, meeting the directrix at Z, PFZ is a right angle.

Hence in the right-angled triangles PMZ, PFZ, since PZ is common, and $PM = PF$, we have $MPZ = FPZ$.

COR. 1. If PFP' is a focal chord, the tangents at its extremities intersect in the directrix.

For since ZFP is a right angle, ZFP' is also a right angle, therefore ZP' also subtends a right angle at F, and is therefore the tangent at P'.

COR. 2. PZP' is a right angle.

For PZ and $P'Z$ bisect the adjacent angles MZF, XZF. Hence *tangents at the extremities of a focal chord intersect at right angles in the directrix.*

COR. 3. If FM cuts PZ in Y, it follows from the triangles PMY, PFY that $MY = YF$, and that the angles at Y are right angles.

Join AY, and since $FY = YM$ and $FA = AX$, AY is parallel to the directrix, and is therefore the tangent at A.

Therefore *the locus of the foot of the perpendicular from the focus on the tangent is the tangent at the vertex.*

COR. 4. Since FYM is perpendicular to the tangent and $FY = YM$, M is called the *image* of the focus in the tangent. It follows that *the locus of the image of the focus in the tangent is the directrix.*

THEOREM 5. SEGMENTS OF THE AXIS.

If NT *is the subtangent,* NG *the subnormal, to prove*

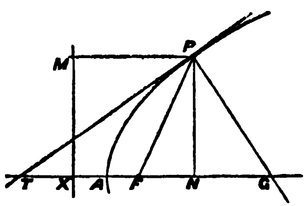
$$NT = 2AN \text{ and } NG = 2AF.$$

Since $\qquad FPT = TPM$ (Th. 4);

and $TPM =$ the alternate angle PTF;

$$\therefore FPT = PTF;$$

$$\therefore FP = FT.$$



And since $$FP = PM = XN,$$

$$\therefore FT = XN,$$

but $$AF = XA,$$

$$\therefore AT = AN,$$

and $$NT = 2AN.$$

Again, since TPG is a right angle, FPG is the complement of FPT, and FGP the complement of FTP;

$$\therefore FGP = FPG \text{ and } FP = FG.$$

$$\therefore FG = FP = PM = XN,$$

and taking away FN,

$$\therefore NG = FX$$
$$= 2AF.$$

THEOREM 6. ORDINATE AND ABSCISSA.

The square of the ordinate is equal to the rectangle contained by the abscissa and the latus rectum.

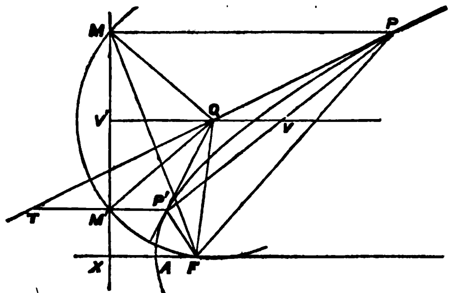
$$(PN^2 = 4AF \cdot AN).$$

Referring to the last figure, since the angle TPG is a right angle,

$$PN^2 = TN \cdot NG$$
$$= 2AN \times 2AF \text{ (by Theorem 5)}$$
$$= 4AF \cdot AN.$$

THEOREM 7. PAIRS OF TANGENTS.

Tangents from any point subtend equal angles at the focus, and have equal projections on the directrix; and the triangles formed by the tangents with the focal distances are similar.



Let QP, QP' be tangents drawn from Q; PM, $P'M'$ perpendiculars to the directrix.

Then by the equal triangles FPQ, MPQ, $FQ = MQ$, and $QMP = QFP$.

Similarly $M'Q = FQ$, and $QM'P' = QFP'$.

$\therefore$ Q is the centre of a circle $MM'F$, and the chord MM' is the projection of PP' on the directrix.

And since $QM = QM'$, $QMM' = QM'M$;

$$\therefore \text{ the angles } QMP,\ QM'P' \text{ are equal.}$$

But since $QMP = QFP$, and $QM'P' = QFP'$,

$$\therefore QFP = QFP';$$

that is, ***tangents subtend equal angles at the focus.***

Again, since QM, QM' are equal, and equally inclined to MM', the diameter through Q will bisect MM', and therefore the projections MV', $M'V'$ of QP, QP' on the directrix are equal.

Again, by joining FM, since FMM', QPM are each complementary to FMP;

$$\therefore FMM' = QPM;$$

$\therefore FPQ = QPM = FMM' = \frac{1}{2} FQM'$ (which is the angle at the centre Q on the same arc FM')

$$= FQP'.$$

Hence the triangles QPF, $P'QF$ are similar.

COR. 1. The diameter through Q bisects PP' in V. For PV, $P'V'$ have equal projections MV', $M'V'$ on the directrix.

COR. 2. If PQ meet $P'M'$ in T,

since $\qquad FQP' = FPQ = QPM = QTP'$

and $\qquad FP'Q = QP'T, \qquad$ (Th. 4),

therefore *the triangles* FP′Q, QP′T *are similar.*

COR. 3. Hence *a pair of tangents can be drawn to a parabola from any external point.*

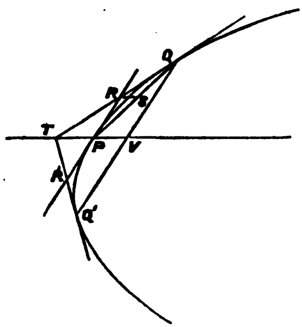
Let Q be the given point; describe a circle with centre Q, and radius QF, to meet the directrix in M, M'; and draw MP, $M'P'$ perpendicular to the directrix to meet the curve in P, P'. Then QP, QP' will be the tangents. For from the equal triangles FPQ, MPQ, the angle FPQ = the angle MPQ; and therefore QP is the tangent at P (Th. 4).

THEOREM 8. DIAMETERS.

A diameter bisects all chords parallel to the tangent at its extremity.

Let PV be a diameter, PR the tangent at P meeting the tangent at Q in R; and let QQ' be parallel to PR.

Then will QQ' be bisected in V.



Let the tangent at Q meet PR in R.

Draw RS parallel to the axis.

Then $QS = SP$ (by Th. 7, Cor. 1), and $\therefore$ $TR = RQ$,

and $\therefore$ $TP = PV$.

Similarly if the tangent at Q' meet VP produced in T', $T'P = PV$, $\therefore$ T and T' are identical, that is, the tangents at QQ' intersect on the diameter through P.

But the diameter through T bisects QQ' (Th. 7);

$\therefore$ the diameter through P bisects all chords parallel to the tangent at P.

COR. $QV = 2PR$; for $QV : RP :: TV : TP$.

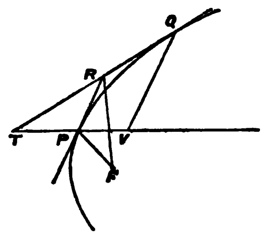
Theorem 9. Oblique Ordinate and Abscissæ.

If QV is the ordinate to the diameter PV,

$$QV^2 = 4FP \cdot PV.$$

For $\quad QV = 2PR;$

and $\therefore QV^2 = 4PR^2:$



let QR meet PV in T; then the triangles FPR, RPT are similar, by Th. 7, Cor. 2,

$\therefore FP : PR :: PR : TP;$

$\therefore PR^2 = FP \cdot TP;$

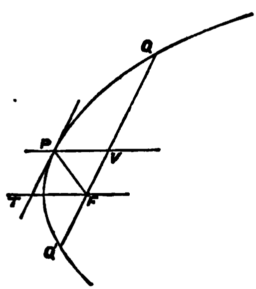
$\therefore QV^2 = 4FP \cdot PT;$

but $\quad PT = PV \quad$ (Th. 8),

$\therefore QV^2 = 4FP \cdot PV.$

Theorem 10. The Parameter.

The parameter of the diameter through P = 4FP.



Let $QVFQ'$ be parallel to PT, the tangent at P;

then $\quad FP = FT = PV$ (Th. 5),

but $\quad QV^2 = 4FP \cdot PV$ (Th. 9)

$= 4FP^2,$

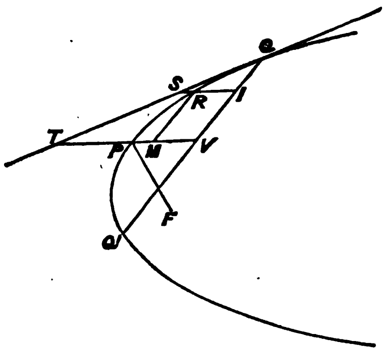
$\therefore QV = 2FP,$

and $\therefore QQ' = 4FP.$

(Omit 11 & 12)

THEOREM 11. SEGMENTS OF DIAMETER MADE BY TANGENT AND CHORD.

If a diameter of a parabola is cut by a chord, and the tangent at the extremity of the chord, the segments of the diameter made by the curve are in the same ratio as the segments of the chord.



Let the diameter SRI meet the chord QQ' in I, the curve in R, and the tangent QT in S, then is

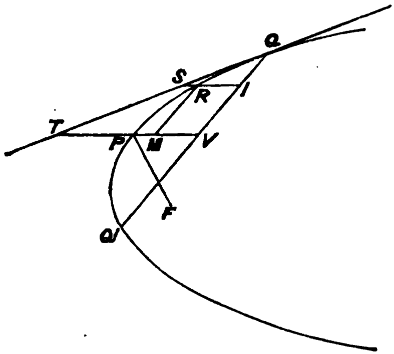
$$SR : RI :: QI : IQ'.$$

Draw TPV the diameter to which QQ' is an ordinate, and draw RM parallel to QQ', to meet PV in M. Join PF.

Then because $QV^2 = 4FP \cdot PV$ (Th. 9)

and $RM^2 = 4FP \cdot PM$, (Th. 9.)

$$\therefore QV^2 - RM^2 = 4FP \cdot MV;$$



but $$QV^2 - RM^2 = QV^2 - IV^2$$
$$= QI \,.\, IQ',$$
and $$MV = RI;$$
$$\therefore\ QI \,.\, IQ' = 4FP \,.\, RI,$$
and $$\therefore\ RI : IQ' :: IQ : 4FP. \qquad (1)$$

Again, by similar triangles

$$SI : IQ :: TV : QV,$$
$$:: 2PV : QV, \qquad \text{(Th. 8.)}$$
$$:: QV : 2FP,$$
(because $QV^2 = 4FP \,.\, PV$)
$$:: QQ' : 4FP,$$
and $$\therefore\ SI : QQ' :: IQ : 4FP. \qquad (2)$$

Therefore combining (1) and (2)

$$RI : IQ' :: SI : QQ'$$
and $$RI : SI :: IQ' : QQ',$$
or $$SR : RI :: QI : IQ'.$$

COR. $QI^2 = 4FP \,.\, SR.$

For $$SR : RI :: QI : IQ';$$

$$\therefore\ 4FP \,.\, SR : 4FP \,.\, RI :: QI^2 : QI \,.\, IQ',$$

but $$4FP \,.\, RI = QI \,.\, IQ';$$

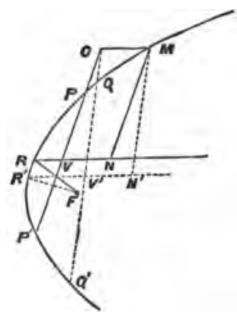
$$\therefore\ QI^2 = 4FP \,.\, SR.$$

This property of the parabola is of great use in the theory of projectiles.

THEOREM 12. SEGMENTS OF INTERSECTING CHORDS.

If two chords of a parabola PP′, QQ′ *intersect in* O, PO . OP′ : QO . OQ′ *in the ratio of the parameters of the diameters which bisect the chords.*

Draw the diameter RV bisecting PP'; draw OM parallel to the axis, and MN parallel to OV.



Then $PO \,.\, OP' = OV^2 - PV^2$

$$= MN^2 - PV^2$$

$$= 4FR \,.\, RN - 4FR \,.\, RV$$

by Th. 9;

$$= 4FR \,.\, VN$$

$$= 4FR \,.\, OM.$$

Similarly $QO \,.\, OQ' = 4FR' \,.\, OM,$

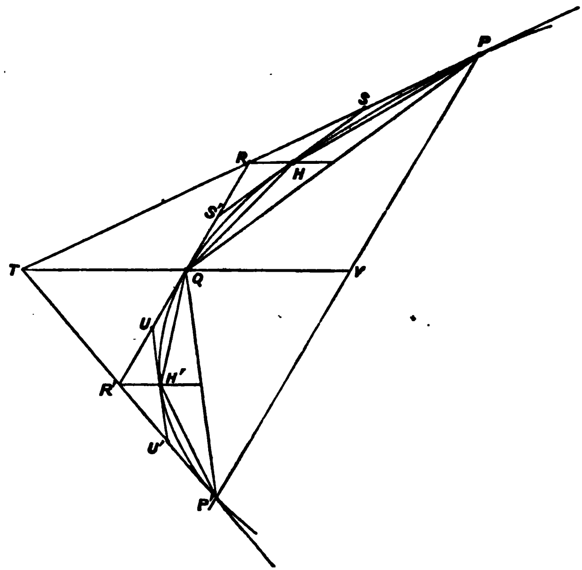
$$\therefore\ PO \,.\, OP' : QO \,.\, OQ' :: 4FR : 4FR',$$

which proves the proposition.

THEOREM 13. AREA OF PARABOLA.

The area of a parabola cut off by any chord is two-thirds of the area of the triangle formed by the chord and the tangents to the parabola at its extremities.

Let PP' be any chord of the parabola PQP', PT, $P'T$ the tangents to the parabola at the extremities of PP'; then will the area included between the curve PQP' and the chord PP' be $\frac{2}{3}$ of the area of the triangle PTP'.



Draw TQV parallel to the axis to meet the curve in Q and the chord in V, and draw the tangent RQR'. Join QP, QP'.

Then, since $TR = \frac{1}{2} TP$, (Th. 8),

the triangle $TQR = \frac{1}{2}$ of the triangle TQP;

and, since $TQ = QV$, (Th. 8),

the triangle TQP = the triangle PQV;

$\therefore$ the triangle $TQR = \frac{1}{2}$ of the triangle PQV.

Similarly the triangle $TQR' = \frac{1}{2}$ of the triangle $P'QV$;

$\therefore$ the triangle $RTR' = \frac{1}{2}$ of the triangle PQP'.

Again, by drawing diameters through R, R', to meet the parabola in H, H', and drawing tangents SS' at H, and UU' at H', and joining PH, HQ, QH', $H'P'$, it may be similarly proved that the triangles SRS', $UR'U'$ are respectively halves of the triangles PHQ, $QH'P'$.

And therefore, by adding, the area $TSS'UU'T$ is half the area $PHQH'P'P$.

And by continuing this process, drawing diameters through S, S', U, U', and drawing tangents at the points where these diameters meet the curve, it is plain that the polygon formed by the tangents outside the curve is *always* half the polygon formed by the chords inside the curve.

And therefore this is true when the number of the sides of the polygon is indefinitely increased.

But in the limit the exterior polygon becomes the area included by the tangents and the curve; and the interior polygon becomes the area included by the chord and the curve;

therefore the exterior area $= \frac{1}{2}$ the interior area;

and therefore the interior area $= \frac{2}{3}$ of the whole area,

$= \frac{2}{3}$ of the triangle PTP'.[1]

Exercises on the Parabola.

1. If FY is the perpendicular from the focus F to the tangent at P, prove that $FY^2 = AF \cdot FP$.

2. If QP, QP' are two tangents to a parabola, F the focus, prove that

$$QF^2 = PF \cdot P'F.$$

3. The tangent at any point cuts the directrix and the latus rectum produced at points equally distant from the focus.

4. To construct a parabola having given two points on the curve, and either the focus or the directrix.

5. To construct a parabola having given the focus, one point, and either one point on the directrix, or one tangent.

6. If Q be any point on the tangent at P, QR, QL perpendicular to the directrix and FP respectively, prove that $$QR = FL.$$

[1] This theorem is due to Archimedes. It was the first instance of the quadrature of a curvilinear area; that is, of finding a rectilineal area (which can be converted into a square) exactly equal to a curvilinear area.

7. The focal distance of a point is greater than, equal to, or less than its distance from the directrix according as the point is outside, on, or inside the parabola.

8. If PM, $P'M'$ are perpendiculars on the directrix from the extremities of a focal chord PFP', prove that MFM' is a right angle.

9. PN, $P'N'$ are the ordinates of the extremities of a focal chord, prove that $PN \times P'N' = \left(\frac{1}{2}\text{ lat. rect.}\right)^2$.

10. Hence prove that $FN \times FN' = XZ^2$.

11. Given two tangents at right angles to one another, and their points of contact, to find the vertex.

12. The chord of contact of two tangents from Q subtends the same angle at the focus, that its projection on the directrix subtends at Q.

13. If a parabola touches three sides of a triangle, its focus will lie on the circle circumscribing the triangle.

14. If QP, QP' are tangents from Q, prove that

$$QP^2 : QP'^2 :: FP : FP'.$$

15. Prove that the lengths of two tangents from any point are as the perpendiculars on them from the focus.

16. Prove that $PG^2 \propto FP$.

17. If $FP \,.\, FP'$ is constant, prove that the locus of the intersection of the tangents at P, P' is a circle.

18. Prove that the circle on FP as diameter touches the tangent at the vertex.

19. Prove that the circle on any focal chord as diameter touches the directrix.

20. A point moves so that its distance from a circle is equal to its distance from a diameter of that circle. Shew that it moves in a parabola.

21. Prove that normals at the extremities of a focal chord intersect on the diameter which bisects the chord.

22. Find the focus and directrix of a parabola that touches four straight lines.

23. If two tangents to a parabola be cut by a third the alternate segments will be proportional.

24. Find the locus of points, such that the sum or difference of their distances from a fixed point or circle and a fixed straight line are given.

25. If a parabola roll on an equal parabola, their vertices having been placed together, the focus of the former will describe the directrix of the latter.

26. As the latus rectum is to the sum of any two ordinates, so is the difference of these ordinates to the difference of the abscissæ. (Th. 6.)

27. Prove Theorem 6 directly from the figure in Th. 1.

28. Any secant through the focus is harmonically divided by the focus and directrix.

29. If from the point of contact of a tangent with a parabola two lines be drawn to the vertices of any two diameters, each to intersect the other diameter; then the line joining these two points of intersection will be parallel to the tangent. (Th. 11.)

CHAPTER II.

THE ELLIPSE AND HYPERBOLA. PROPERTIES COMMON TO BOTH CURVES.

THE ellipse and hyperbola are *central conic sections*, that is they have a centre, in which, as will appear, every chord that passes through it is bisected. The Parabola has no centre. Hence the ellipse and hyperbola may be conveniently studied together, many of their properties being identical.

In the present chapter the proofs of the properties common to the ellipse and hyperbola are given, with figures of both curves.

In the next chapter some properties are given which are either different for the two curves, or are most easily obtained by different modes of proof.

THEOREM I.

An Ellipse has the following properties:

(1) *There are two points in its plane such that the sum of their distances from any point on the curve is constant.*

(2) *The ratio of the distances of every point on the curve from a fixed point and fixed straight line in its plane is constant.*

(3) *There exists a line in the plane of the ellipse such that the ordinates of the ellipse to abscissæ measured along this line are to the ordinates of the circle described on this line as diameter in a constant ratio.*

A Hyperbola has the following properties:

(1) *There are two points in its plane such that the difference of their distances from every point on the curve is constant*[1].

(2) *The ratio of the distances of every point on the curve from a fixed point, and a fixed straight line in its plane, is constant.*

(3) *There exists a line in the plane of the hyperbola such that the ordinates of the hyperbola to abscissæ measured along this line produced, are to tangents drawn from the feet of these ordinates to the circle described on this line as diameter in a constant ratio.* (Vid. fig. of Th. 7.)

See figure at the end of the book.

Let S be the vertex of a right circular cone of which SOO' is the axis, and let the plane of the paper contain the axis and the generators SVU, $SV'U'$; and let any plane perpendicular to the plane of the paper, and intersecting it in AA', obliquely to the axis, cut the surface in the ellipse or the hyperbola APA'. (Vid. p. 45.)

Since the plane of the paper is perpendicular to the plane APA', the centres of the spheres which touch the

[1] The proof of (1) is the same as that in the corresponding theorem on the ellipse, the sum of the distances being changed into their difference.

The proof of (2) is also the same as in the ellipse.

The proof of (3) is also the same, the ordinate from N to the circle being changed into the tangent from N to the same circle.

plane APA' along the line AA' will be in the plane of the paper. (IV. 35, Cor. 3.)

Hence, if O, O' are the centres of circles which touch AA' and the generators SA, SA' (that is, centres of the inscribed and escribed circles of the triangle SAA'), spheres may be described with centres O, O' to touch the plane APA' in two points F, F' on the line AA', and to touch the cone along two circles whose planes are perpendicular to the axis, that is, along VKV', and $UK'U'$. (IV. 36.)

Let P be any point on the ellipse; $SKPK'$ the generator passing through P, touching the spheres in K, K'.

Join FP, $F'P$.

Then (1) in the ellipse $FP+F'P=$ a constant.

For $FP=KP$, being tangents to a sphere whose centre is O from the same point P. (IV. 36, Cor.)

And $F'P=K'P$ for a similar reason. $2AA' = A'F' + AF' + A'F + FA$

Therefore $= A'U' + A'V' + AU + AV$

$$FP+F'P=KP+K'P=KK'=SK'-SK=VU,$$ $= V'U' + UV$

which is constant for all positions of P.

$\therefore AA' = UV$

The points F, F' are called the *foci*.

$AF = AV = UV - AV' = UV - AF' = AA' - AF'$

It follows from well-known theorems in plane geometry on the inscribed and escribed circles that

$$VU=AA', \text{ and that } AF=A'F', \text{ and } FF'=SA'-SA.$$ $\therefore AF = A'F'$

(2) Let the plane of the circle $V'KV$ intersect the plane APA' in the line XM, which will therefore be at right angles to the plane of the paper. (IV. 18, Cor.)

From P draw PM perpendicular to XM.

Then shall PF be to PM in a constant ratio.

Draw a plane through P perpendicular to the axis of the cone, intersecting APA' in PN, which will therefore be at right angles to AA' (IV. 18, Cor.), and meeting the cone in the circle WPW'. (Vid. p. 44.)

Then $PF = PK = VW$,

and $PM = NX$;

$\therefore PF : PM :: VW : NX$,

$:: VA : AX$, since XV is parallel to NW,

$:: AF : AX$;

that is, $PF : PM$ in a constant ratio for all positions of P.

Similarly, if the plane $UK'U'$ intersect APA' in $X'M'$, and PM' is drawn perpendicular to $X'M'$,

$$PF' : PM' :: W'U' : NX',$$
$$:: A'U' : A'X',$$
$$:: A'F' : A'X'.$$

The lines XM, $X'M'$ are called directrices, and F, F' the corresponding foci.

It must be observed that

$$AF : AX :: AV : AX :: VU : XX',$$
$$:: V'U' : XX' :: A'U' : A'X' :: A'F' : A'X'.$$

The ratio $PF : PM$ is called the eccentricity of the ellipse, and is generally denoted by the letter e.

e is less than 1 in the ellipse, since AV is then less than AX; and e is greater than 1 in the Hyperbola, since AV is then greater than AX.

Also, since $AF = A'F'$, therefore also $AX = A'X'$.

(3) If WPW' is the circular section through P, draw AB, $A'B'$ parallel to VV'.

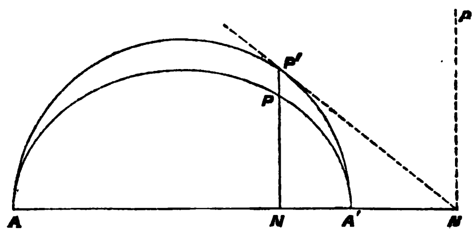
Then $$PN^2 = WN \, . \, W'N;$$

but $$WN : AN :: A'B' : AA',$$

and $$W'N : A'N :: AB : AA';$$

$$\therefore \; WN \, . \, W'N : AN \, . \, A'N : AB \times A'B' :: AA'^2;$$

$\therefore$ PN^2 is to $AN \, . \, A'N$ in a constant ratio.



But if on AA' as diameter a circle were described, and $P'N$ were the ordinate to it through N, $P'N^2 = AN \times A'N$.

Therefore $PN^2 : P'N^2$, or $PN : P'N$, is a constant ratio.

The ellipse therefore has this property, that its ordinate bears a constant ratio to the corresponding ordinate of a circle described on AA' as diameter.

This circle is called the *auxiliary* circle, and the points P, P' are called *corresponding* points.

Both curves are from their mode of construction symmetrical with respect to AA', and since they may be described from either focus and directrix, they must also be symmetrical with respect to an axis bisecting AA' at right angles.

Hence every chord through the intersection of the axes will be bisected in that point, and is called a *diameter.*

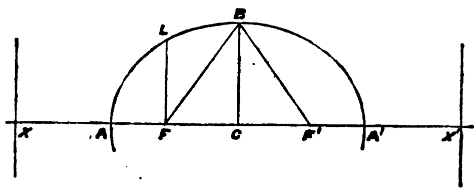
The ellipse will be a closed curve, and the hyperbola will consist of two infinite branches, as may be seen in the figures of Theorem 2.

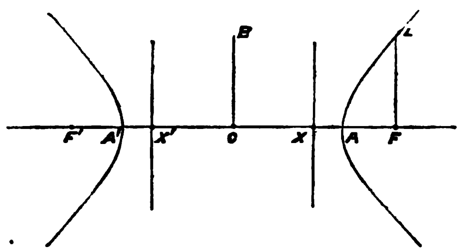
AA' is called the *major* axis or *transverse axis.*

THEOREM 2. SEGMENTS OF THE AXIS.

If A, A' *are the vertices of a central conic,* F, F' *the foci,* X, X' *the feet of the directrices,* C *the middle point of* FF', *then*

$$AF : AX :: CF : CA :: CA : CX.$$





For $$AF : AX :: A'F : A'X,$$
by property (2) of the central conic;

$$\therefore AF : A'F :: AX : A'X;$$

$$\therefore\ AF : FF' :: AX : AA';$$

$$\therefore\ AF : AX :: FF' : AA'$$

$$:: CF : CA.$$

For a similar reason, $AF : AA' :: AX : XX'$,

$$\therefore\ AF : AX :: AA' : XX'$$

$$:: CA : CX.$$

COR. 1. $CF \cdot CX = CA^2$.

COR. 2. If CB is drawn at right angles to AA' to meet the ellipse in B; then $BC^2 = AF \cdot A'F$.

For in the ellipse, since $FB = F'B$, and

$$FB + F'B = AA';\ \therefore\ FB = AC,$$

$$\therefore\ AF \cdot A'F = FB^2 - FC^2,$$

$$= BC^2.$$

$FB^2 - FC^2 = (FB + FC)(FB - FC)$
$A'C \quad AC$
$= A'F \cdot AF$

In the hyperbola a line BC at right angles to AA' through its middle point is taken, such that

$$BC^2 = AF \cdot A'F,$$

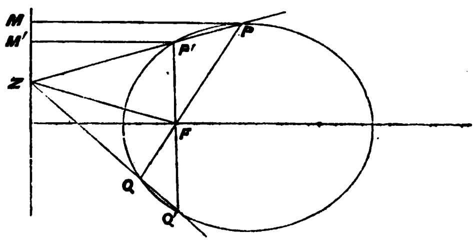
$$\text{or} \quad = FC^2 - AC^2,$$

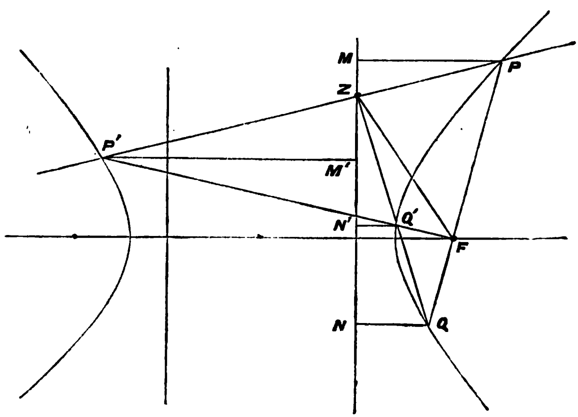
and $2BC$ is called the *minor axis*, or the *conjugate axis* of either ellipse or hyperbola.

Theorem 3. The Secant and Directrix.

If in a central conic a secant PP′ *meet the directrix in* Z, *and* F *is the corresponding focus,* FZ *is the exterior bisector of the angle* PFP′, *or of its supplement.*

Draw PM, $P'M'$ perpendicular to the directrix.





Then $$FP : PM :: FP' : P'M',$$

by a property of a central conic; (Th. 1.)

$$\therefore FP : FP' :: PM : P'M',$$
$$:: PZ : P'Z,$$

by similar triangles; therefore FZ bisects the exterior or interior angle of the triangle PFP'.

It will be observed that in the hyperbola FZ is the bisector of the exterior or interior angle of the triangle PFP', according as the secant meets one branch only or both branches of the curve.

COR. 1. *If* PQ, P'Q' *be focal chords,* QQ' *and* PP' *intersect on the directrix.*

For PP', QQ' both meet the directrix where it is cut by the bisector of $P'FQ$.

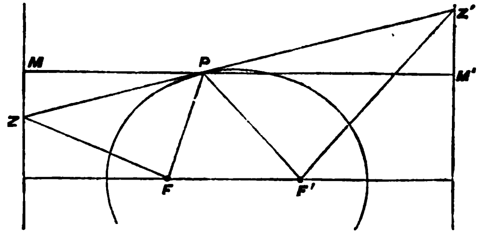
COR. 2. P'Q, PQ' *also intersect on the directrix in a point* Z' *by* Cor. 1; *and* ZFZ' *is a right angle, since* FZ *and* FZ' *are bisectors of adjacent supplementary angles.*

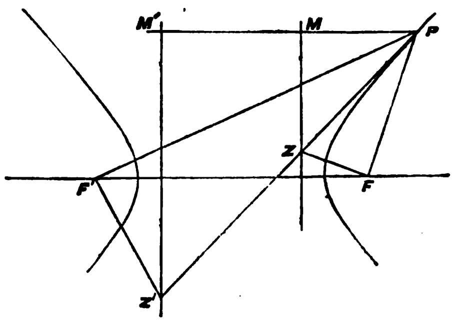
COR. 3. *The tangent being the limiting position of the secant, when the points of intersection approach each other, it follows that when the secants* PP', QQ' *become tangents, the tangents at the extremities of a focal chord intersect in the directrix and subtend right angles at the focus.*

THEOREM 4. THE TANGENT IN A CENTRAL CONIC.

The tangent in a central conic makes equal angles with the focal distances.

Let ZPZ' be the tangent at P, meeting the directrices in Z, Z'.





Since the tangent at P is the limiting position of the secant PP' when P' moves up to P,

FZ is at right angles to FP. (Th. 3, Cor. 3.)

Similarly $F'Z'$ is at right angles to $F'P$.

And $\therefore FP : F'P :: PM : PM'$, (Th. 1.)

and $PM : PM' :: PZ : PZ'$ by similar triangles;

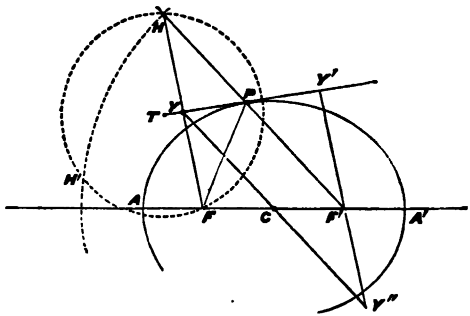
$\because FP : PZ :: F'P : PZ'$.

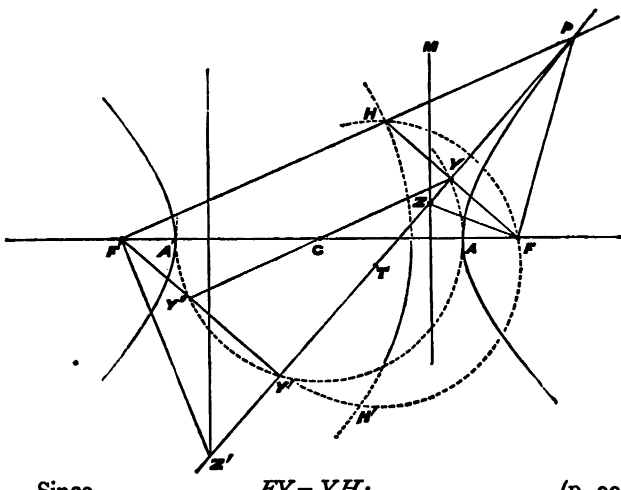
Therefore the right-angled triangles PFZ, $PF'Z'$ have the sides about one of the other angles proportionals;

therefore they are similar;

and therefore $FPZ = F'PZ'$.

COR. 1. *If Y is the foot of the perpendicular from the focus on the tangent, H the image of the focus in the tangent, the loci of Y, H are circles.*





Since $FY = YH$; (p. 99)

and the angles at Y are right angles;

$$\therefore FP = HP, \text{ and } FPY = HPY;$$

but $FPY = F'PY'$ by the theorem;

$\therefore$ $HPY = F'PY'$ and HPF' or $F'HP$ is a straight line; but $F'H = F'P \pm PH$, the upper sign being taken for the ellipse, and the lower for the hyperbola;

$$= F'P \pm FP$$

$$= AA' = \text{constant.}$$

Therefore the locus of H is a circle described round F' as centre with radius equal AA'.

This is called a *director* circle.

Since the tangent bisects HF at right angles it follows that if T be any point on the tangent,

$$TH = TF.$$

Again, to find the locus of Y, join YC.

Then, since $\quad FY = YH$ and $FC = CF'$;

$$\therefore\ FY : YH :: FC : CF';$$

and $\therefore$ YC is parallel to $F'H$, and $= \frac{1}{2} F'H$

$$= CA;$$

therefore the locus of Y is the *auxiliary* circle. (Th. 1.)

Cor. 2. *Hence a tangent may be drawn to the conic from any point.*

Draw a circle with centre T and radius TF, to cut the director circle whose centre is F' in H, H', and join HF', cutting the curve in P, and join TP. TP is a tangent. For, since $F'H = AA' = F'P \pm FP$, $\therefore$ in the triangles FPT, HPT, the three sides are respectively equal, and $\therefore$ TP makes equal angles at P with the focal distances of

P; and $\therefore$ TP is the tangent at P. The other tangent is similarly found by joining $H'F'$.

COR. 3. *If* F'Y' *is the perpendicular from the other focus on the tangent,*

$$FY \,.\, F'Y' = AF \,.\, A'F = BC^2.$$

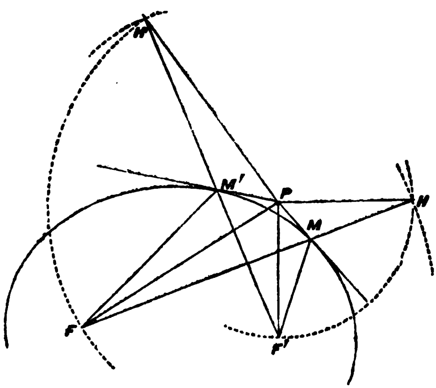
Produce YC to meet $Y'F'$ in Y'', then, since Y' is a right angle, YCY'' is a diameter of the auxiliary circle: and $CY'' = CY$, and $\therefore$ from the triangles CFY, $CF'Y''$, $F'Y'' = FY$;

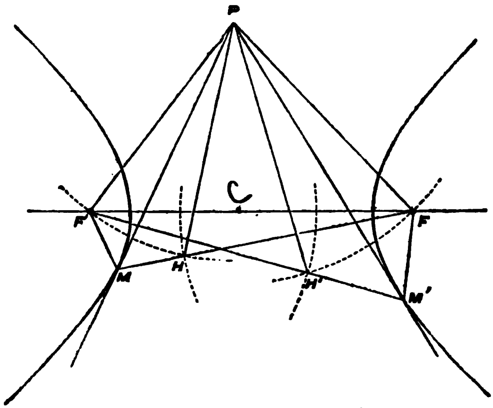
$$\therefore\ FY \,.\, F'Y' = F'Y'' \,.\, F'Y' = F'A \,.\, F'A' = BC^2$$

(Th. 2, Cor. 2).

THEOREM 5. PAIR OF TANGENTS.

The tangents from P *to a central conic make equal angles with the focal distances of* P, *and subtend equal or supplementary angles at either focus.*





Let H, H' be the images of F', F, in PM, PM' respectively found as in Th. 4, Cor. 2, and let PM, PM' be the tangents.

Then $FH = AA' = F'H'$, and $PH = PF'$, $PH' = PF$ by Th. 4.

Therefore the triangles FPH, $F'PH'$ are equal in all respects;

$$\therefore \text{ the angle } FPH = F'PH',$$

and $$\therefore\ HPF' = H'PF;$$

$$\therefore\ F'PM = FPM';$$

that is, the tangents make equal angles with the focal distances.

Also $PF'M = PHM$; and PHM is equal or supplementary to PHF, that is to $PF'M'$: or the tangents subtend equal or supplementary angles at the focus.

COR. *If the tangents from* P *include a right angle, the locus of* P *is a circle.*

For if MPM' is a right angle, so is also FPH, since $MPH = M'PF$,

$$\therefore\ FP^2 + F'P^2 = FP^2 + PH^2 = FH^2 = \text{const.}$$

But $$FP^2 + F'P^2 = 2CP^2 + 2CF^2;$$

and since $$2CP^2 = FH^2 - 2CF^2,$$
$$= 4AC^2 - 2CF^2,$$

and $$CF^2 = AC^2 \mp BC^2 \qquad \text{(Th. 2, Cor. 2)},$$
$$\therefore\ CP^2 = AC^2 \pm BC^2,$$

and the locus of P is a circle.

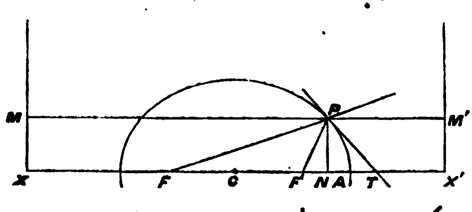
THEOREM 6. THE SUBTANGENT ON THE TRANSVERSE AXIS.

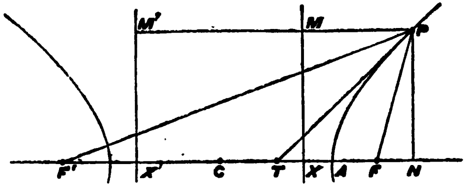
In a central conic if the tangent at P *meet the transverse axis in* T,

$$CT \cdot CN = CA^2.$$

Since PT bisects the angle at P between the focal distances,

$$FT : F'T :: FP : F'P$$
$$:: PM : PM'$$
$$:: XN : X'N;$$





$$\therefore\ FT+F'T : FT\sim F'T :: XN+X'N : XN\sim X'N,$$

$$\text{or } 2CT : 2CF :: 2CX : 2CN;$$

$$\therefore\ CT.CN = CF.CX$$

$$= CA^2. \qquad \text{(Th. 2.)}$$

Cor. $\qquad TA : TN :: TC : TA'.$

Corresponding Points and Lines; the Auxiliary Circle.

In the ellipse it was shewn, Theorem 1, that the ordinates to the axis are all less than the ordinates to the same abscissa of the auxiliary circle in the same ratio; i. e. $PN : P'N$ in a constant ratio.

The points P, P' are *corresponding points*; PQ, $P'Q'$ are called *corresponding lines*.

If $B'BC$ is drawn an ordinate through C,

$$PN : P'N :: BC : B'C$$

$$:: BC : AC,$$

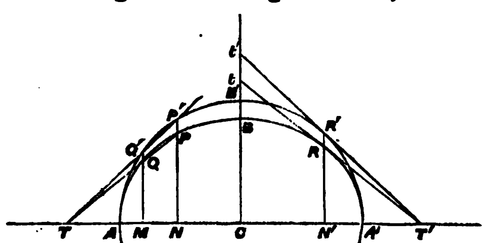
where BC is the semi-axis minor, and AC the semi-axis major.

Lemma. *Corresponding lines in the ellipse intersect on the axis.*

Let PQ, $P'Q'$ be corresponding lines, and let PQ meet the axis in T. Then T is determined by the ratio

$$MT : NT :: QM : PN,$$

but $\qquad QM : PN :: Q'M : P'N;$



and therefore the point where $P'Q'$ meets the axis is determined by the same ratio. Therefore PQ, $P'Q'$ intersect on the axis.

COR. *Tangents at corresponding points intersect on the axis.*

m. 7

THEOREM 7. ORDINATE AND ABSCISSA.

In a central conic

$$PN^2 : AN \,.\, A'N :: BC^2 : AC^2.$$

(1) In the ellipse (using the figure in the Lemma),

Let P, P' be corresponding points, then

$$PN^2 : P'N^2 :: BC^2 : AC^2.$$

But $$P'N^2 = AN \,.\, A'N;$$

$$\therefore PN^2 : AN \,.\, A'N :: BC^2 : AC^2.$$

(2) In the hyperbola.

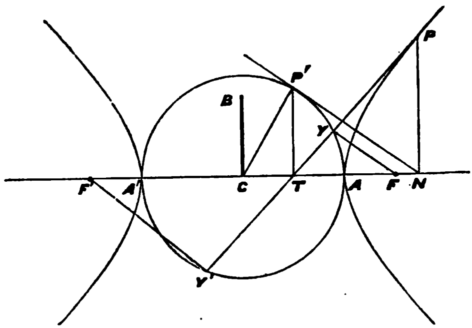
Let PN be the ordinate, NP' the tangent to the auxiliary circle from N, and let the tangent from P meet the axis in T.

Then, since

$$CT \,.\, CN = CA^2 = CP'^2, \qquad \text{(Th. 6.)}$$

T is the foot of the perpendicular from P' on the axis.

Draw FY, $F'Y'$ perpendicular to the tangent.



Then, by similar triangles PNT, FYT,

$$PN : FY :: TN : TY,$$

and similarly, $PN : F'Y' :: TN : TY'$;

$$\therefore PN^2 : FY \cdot F'Y' :: TN^2 : TY \cdot TY';$$

$$\therefore PN^2 : BC^2 :: TN^2 : P'T^2 \text{ (Th. 4, Cor. 3.)}$$

$$:: P'N^2 : CP'^2$$

$$:: AN \cdot A'N : AC^2;$$

$$\therefore PN^2 : AN \cdot A'N :: BC^2 : AC^2.$$

Cor. *The latus rectum in a central conic is a third proportional to the axes major and minor.*

For (using the figure in Theorem 2),

$$LF^2 : AF \cdot A'F :: BC^2 : AC^2$$

by the Theorem just proved;

$$\therefore LF^2 : BC^2 :: BC^2 : AC^2,$$

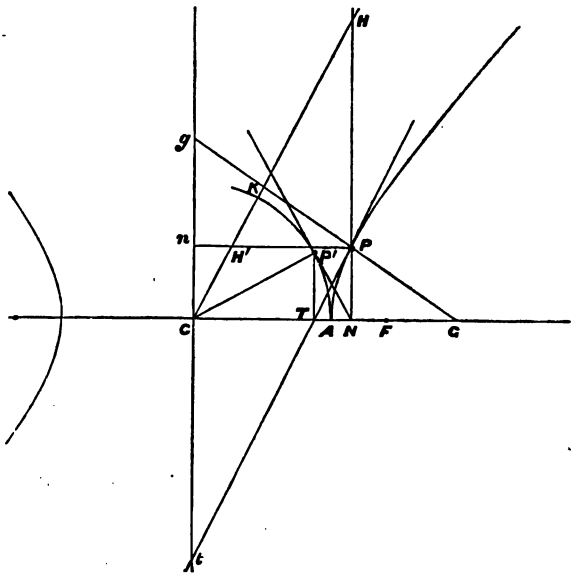
$$\text{and } \therefore LF : BC :: BC : AC.$$

THEOREM 8. SUBTANGENT ON MINOR AXIS.

In a central conic, if the tangent at P *meet the conjugate axis in* t, *and* Pn *is perpendicular to that axis,*

$$Cn \,.\, Ct = BC^2.$$

Since $$Cn = PN,$$



$$\therefore\ Cn : CN :: PN : CN,$$

but $$Ct : CT :: PN : TN;$$

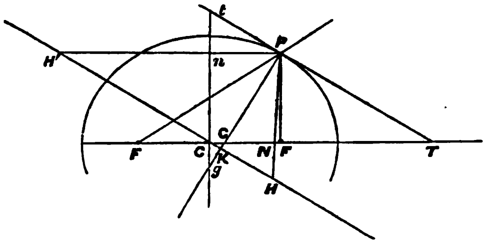
and $$\therefore\ Cn\,.\,Ct : CT\,.\,CN :: PN^2 : TN\,.\,CN.$$

But $TN\,.\,CN = P'N^2$, where P' corresponds to P;

$$\therefore\ Cn\,.\,Ct : AC^2 :: PN^2 : P'N^2$$

$$:: BC^2 : AC^2;$$

$$\therefore\ Cn\,.\,Ct = BC^2.$$



Theorem 9. The Normal.

If in a central conic the normal meets the axes major and minor in G, g, *and* CK *is perpendicular to the normal, then* $PG.PK=BC^2$, $Pg.PK=AC^2$, *and* $CG : CN :: CF^2 : AC^2$.

Using the figures of Theorem 8,

Draw PN, Pn perpendicular to the axes, and produce them to meet CK, which is parallel to the tangent at P, in H, H'.

Draw TPt the tangent at P.

Then, since a circle may be described round $KGNH$, the angles at K and N being right angles,

$$PG.PK=PN.PH=Cn.Ct=BC^2. \quad \text{(Th. 8.)}$$

Again, since a circle may be described round $H'nKg$,

$$Pg.PK=Pn.PH'=CN.CT=AC^2;$$

$$\therefore \text{ also } PG : Pg :: BC^2 : AC^2;$$

but
$$GN : CN :: GN : Pn,$$
$$:: PG : Pg,$$

and
$$\therefore GN : CN :: BC^2 : AC^2;$$
$$\therefore CG : CN :: AC^2-BC^2 : AC^2$$
$$:: CF^2 : AC^2.$$

CHAPTER III.

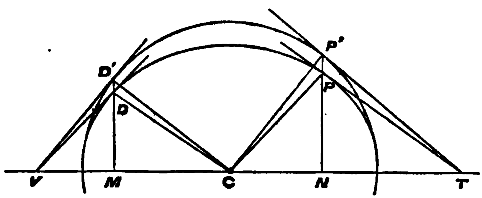
THE ELLIPSE AND HYPERBOLA CONTINUED.

Def. A diameter is said to be *conjugate* to another when it is parallel to the tangent at the extremity of the latter.

Def. An ordinate to a diameter is the line drawn parallel to the tangent at the extremity of that diameter.

THEOREM 10. PROPERTIES OF THE ELLIPSE.

In the ellipse if CP *is conjugate to* CD, *then is* CD *conjugate to* CP.



Draw the tangents TP, VD at P, D to the ellipse, and at the corresponding points on the auxiliary circle. Then CP is given parallel to DV; and it is required to prove CD parallel to PT.

By similar triangles we have

$$VM : MD :: CN : NP;$$

$$\therefore\ VM : MD' :: CN : NP';$$

therefore VD' is parallel to CP',

$$\text{and } \therefore\ P'CD' = CD'V \text{ is a right angle};$$

$$\therefore\ P'CD' = TP'C;$$

$$\text{and } \therefore\ TP' \text{ is parallel to } CD';$$

$$\therefore\ D'MC,\ P'NT \text{ are similar},$$

$$\text{and } \therefore\ D'M : MC :: P'N : NT,$$

$$\text{and } \therefore\ DM : MC :: PN : NT,$$

$$\text{and } \because \text{ the triangles } DMC,\ PNT \text{ are similar},$$

and $\therefore$ CD is parallel to PT.

Cor. 1. The triangles $P'NC$, CMD' are equal in all respects,

$$\text{and } \therefore\ CM^2 + CN^2 = P'N^2 + CN^2 = CP'^2 = AC^2.$$

Cor. 2. $$DM : CN :: BC : AC.$$

Cor. 3. $$DM^2 + PN^2 = BC^2,$$

for $$DM^2 : CN^2 :: BC^2 : AC^2,$$

and $$PN^2 : CM^2 :: BC^2 : AC^2;$$

$$\therefore\ DM^2 + PN^2 : CN^2 + CM^2 :: BC^2 : AC^2,$$

and $$CN^2 + CM^2 = AC^2;$$

$$\therefore\ DM^2 + PN^2 = BC^2.$$

Cor. 4. $$CP^2 + CD^2 = AC^2 + BC^2.$$

$Cm^2 + \overline{D}m^2 = \overline{CD}^2$

$\overline{C}n^2 + \overline{P}n^2 = \overline{PC}^2$

$\overline{CD}^2 + \overline{PC}^2 = \overline{AC}^2 + \overline{BC}^2.$

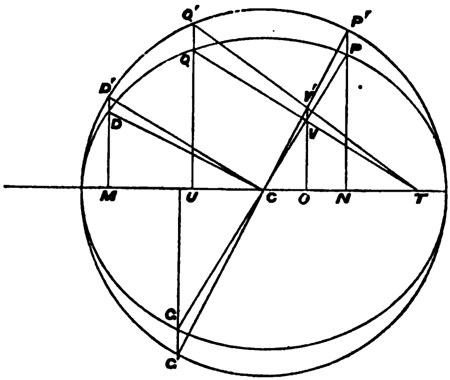
THEOREM 11. OBLIQUE ORDINATES AND ABSCISSÆ.

In the ellipse if QV *is the ordinate to the diameter* PVG, *and* CD *is conjugate to* CP,

$$QV^2 : PV \cdot VG :: CD^2 : CP^2.$$

Let P', Q', D' be the corresponding points to P, Q, D; join CP', and let the ordinate of V meet CP' in V', so that

$$OV : OV' :: NP : NP' :: UQ : UQ'.$$



Then, by a former Lemma, p. 128, QV, $Q'V'$ intersect in the axis at some point T.

And by similar triangles (as in Th. 10) $Q'V'$ may be proved to be parallel to CD', and therefore at right angles to CP';

$$\therefore\ Q'V'^2 = P'V' \cdot V'G';$$

but $$QV^2 : Q'V'^2 :: CD^2 : CD'^2,$$

and $$PV : P'V' :: CP : CP',$$

and $$VG : V'G' :: CP : CP';$$

$$\therefore PV . VG : P'V' . V'G' :: CP^2 : CP'^2,$$

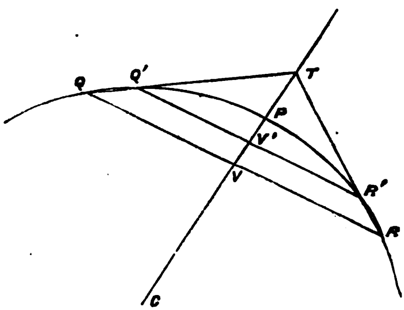
and $\therefore$ since $Q'V'^2 = P'V' . V'G$ and $CD'^2 = CP'^2$,

$$\therefore QV^2 : PV . VG :: CD^2 : CP^2.$$

COR. 1. *Hence all chords of an ellipse are bisected by the diameter to which they are ordinates; and conversely the line that bisects a system of parallel chords is a diameter.*

COR. 2. *If the ordinate and tangent at* Q *meet the diameter in* V, T *respectively,*

then $$CV . CT = CP^2.$$



Let QVR, $Q'V'R'$ be parallel chords in a central conic, bisected by their conjugate diameter CP in V, V'.

Let QQ' meet CP in T;

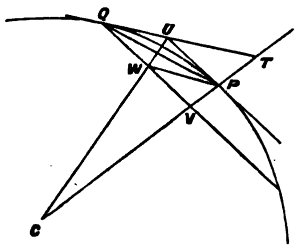
then since $$VT : V'T :: QV : Q'V'$$
$$:: RV : R'V',$$

$\therefore RR'$ also meets CP in T.

Hence when the chords move up to one another, and QT, RT become tangents at Q, R, the tangents at the extremities of a chord intersect on the diameter to which the chord is conjugate.

Hence also the line that joins the middle point of a chord with the point of intersection of the tangents at its extremity passes through the centre.

Now let QV be an ordinate to CP; QT the tangent at its extremity: then will $CV \cdot CT = CP^2$.



Draw PU the tangent at P to QT in U, and PW parallel to QT to meet QV in W.

Then $QUPW$ is a parallelogram, and therefore UW bisects QP; and therefore UW passes through C.

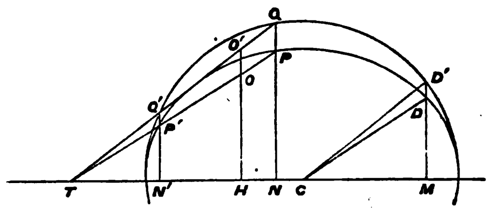
Therefore $$CV : CP :: CW : CU$$
$$:: CP : CT,$$
$$\text{or } CV \cdot CT = CP^2.*$$

* This proof is due to the Rev. C. Taylor, of St John's College, Cambridge, and is inserted by his permission.

THEOREM 12. RECTANGLES CONTAINED BY THE SEGMENTS OF INTERSECTING CHORDS.

If two chords of an ellipse intersect one another, the rectangles contained by the segments of the chords are proportional to the squares of the diameters parallel to them.

Let POP' be one of the chords through O, CD the parallel semidiameter. Let PP' meet the axis in T, and take Q, O', Q', D' corresponding points tc P, O, P', D.



Then QQ' passes through T, and is parallel to CD'.

And since by parallelism

$$PO : QO' :: OP' : O'Q' :: CD : CD',$$

$$\therefore PO . OP' : QO' . O'Q' :: CD^2 : CD'^2,$$

$$\text{or } PO . OP' : CD^2 :: QO' . O'Q' : CD'^2.$$

But if any other chord ROR' were drawn through O, and CS were its parallel semidiameter, then $QO' . O'Q'$, and CD'^2 would remain unaltered, by a property of the circle,

and $\therefore PO . OP' : CD' :: RO . OR' : CS^2,$

which proves the proposition.

COR. *If a circle intersect an ellipse in the points* P, P′, R, R′, *the chords* PP′, RR′ *must be equally inclined to the axis.*

For if PP', RR' intersect in O, $PO \cdot OP' = RO \cdot OR'$ from the circle;

$\therefore CD^2 = CS^2$, or the diameters parallel to the chords are equal, and therefore equally inclined to the axis.

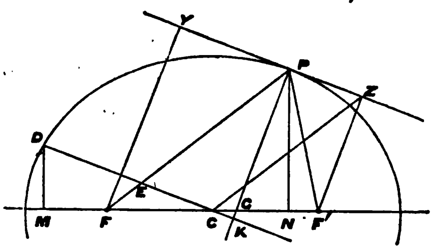
Obs. If one or both of the chords becomes a tangent the rectangle contained by the segments of the chords becomes the square of the tangent.

THEOREM 13.

In the ellipse

$$PK \cdot CD = AC \cdot BC.$$

Draw DM and PN the ordinates to D, P.



Since $\quad PK \cdot PG = BC^2$, (Theorem 9),

$$\therefore PK : BC :: BC : PG.$$

But by similar triangles PGN, DMC,

$$PN : CM :: PG : CD.$$

But $\quad PN : CM :: BC : AC$, (Theorem 10, Cor. 2.)

$$\therefore BC : AC :: PG : CD.$$

But $\quad PK : BC :: BC : PG$,

$\therefore$ ex æquali, $PK : AC :: BC : CD$,

and $\therefore PK \cdot CD = AC \cdot BC.$

Cor. *Hence the areas of all parallelograms formed by tangents at the extremities of conjugate diameters are equal.*

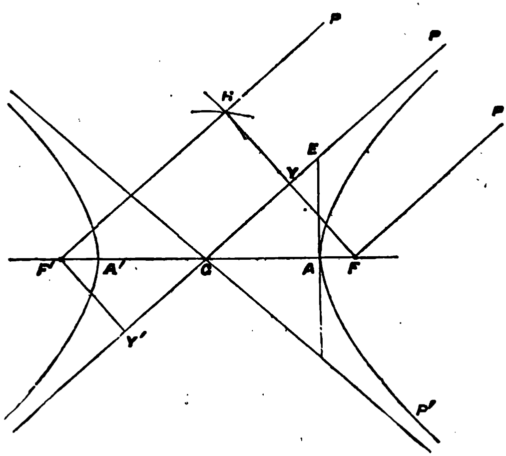
$$\text{For the area} = 4PK \,.\, CD$$
$$= 4AC \,.\, BC.$$

Theorem 14. Properties of the Hyperbola. Asymptotes.

Def. A hyperbola whose asymptotes are the same as that of the given hyperbola, and whose conjugate and transverse axes are the transverse and conjugate axes of the latter, is said to be *conjugate* to the latter hyperbola.

Def. A diameter of one hyperbola is said to be *conjugate* to a diameter of the other when it is parallel to the tangent at the extremity of the latter.

Tangents drawn to a hyperbola from its centre meet the curve at an infinite distance from the centre.



To draw tangents from C (by the construction given in Theorem 4, Cor. 2), describe the director circle with centre F', and a circle with centre C, and radius CF, to intersect the former in H.

Then, since $CF = CH = CF'$, FHF' is a right angle.

Draw CY perpendicular to FH, and therefore parallel to $F'H$, and bisecting FH. Then (by Theorem 4), CY touches the curve at the point where CY and $F'H$ intersect.

But in this case CY and $F'H$ are parallel, or meet the curve at an infinite distance.

Therefore the tangent from the centre meets the curve at an infinite distance.

This tangent is called an *asymptote*, being a line which never meets the curve, though, as will be shewn in the next theorem, it continually approaches to it.

From the symmetry of the curve it is plain that a line equally inclined to the axis on the other side of it is an asymptote to AP', and that these asymptotes produced through the centre are asymptotes to the other branch of the hyperbola.

COR. 1. *The asymptote passes through the intersection of the directrix and the auxiliary circle.*

For, since $CY = \frac{1}{2} F'H = CA$, Y is on the auxiliary circle;

And, since YFP is a right angle, Y is on the directrix, FP being drawn to the point of contact (Theorem 3, Cor. 3).

COR. 2. *If* AE *is drawn to touch the hyperbola at* A, *and meet the asymptote in* E, AE = BC.

For the triangles AEC, YFC are equiangular and have

$$CY = CA, \therefore AE = FY.$$

the 4. l 3,

But $$FY \times F'Y' = BC^2,$$

$$\therefore FY = BC,$$

and $$\therefore AE = BC.$$

It follows that the asymptotes are the diagonals of a rectangle whose sides are the axes, and which touch the vertices of the hyperbola and its conjugate at their middle points.

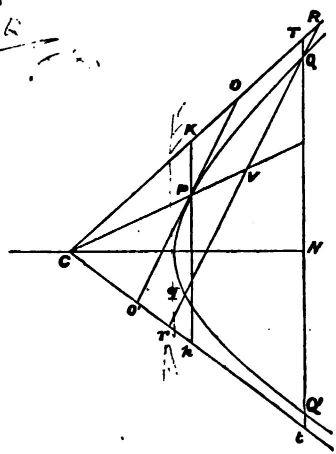
THEOREM 15. SECANTS OF THE CURVE AND ASYMPTOTES.

If TQQ′t, *perpendicular to the axis, cut the asymptotes in* T, t *and the curve in* Q, Q′, *then will*

$$TQ \cdot Qt = BC^2.$$

Let $TQQ't$ meet the axis in N, then $TQ \cdot Qt = TN^2 - QN^2$, and since $TN^2 : CN^2 :: AE^2 : AC^2$ by similar triangles,

$$:: BC^2 : AC^2;$$



and also $QN^2 : AN \,.\, A'N :: BC^2 : AC^2$, (Th. 7.)

$$\therefore\ TN^2 : QN^2 :: CN^2 : AN \,.\, A'N$$
$$:: CN^2 : CN^2 - CA^2.$$
$$\therefore\ TN^2 - QN^2 : QN^2 :: AC^2 : CN^2 - CA^2,$$
$$\therefore\ TN^2 - QN^2 : AC^2 :: QN^2 : CN^2 - CA^2$$
$$:: BC^2 : AC^2, \qquad \text{(Th. 7.)}$$
$$\therefore\ TN^2 - QN^2 = BC^2,$$
$$\text{and}\ \therefore\ TQ \,.\, Qt = BC^2.$$

Hence as N moves away from C and Qt becomes greater, TQ becomes less. That is, the line CE perpetually approaches the curve but never meets it, and is therefore called an *asymptote.*

THEOREM 16.

If OPO′ *be a tangent at* P, *meeting the asymptotes in* O, O′, *and* RQqr *a parallel secant, then will* PO = PO′, RQ = qr, *and* RQ . Qr = PO².

Using the figure of the preceding Theorem, draw $TQQ't$, Kpk perpendicular to the axis.

Then since $RQ : QT :: PO : PK$,

and $Qr : Qt :: PO' : Pk$,

$$\therefore RQ \,.\, Qr : QT \,.\, Qt :: PO \,.\, PO' : PK \,.\, Pk,$$

but $QT \,.\, Qt = PK \,.\, Pk = BC^2$,

$$\therefore RQ \,.\, Qr = PO \,.\, PO'.$$

Hence $\quad RQ \,.\, Qr = Rq \,.\, qr,$

$$\therefore\ RQ \,.\, Qq + RQ \,.\, qr = RQ \,.\, qr + Qq \,.\, qr\,;$$

$$\therefore\ RQ = qr\,;$$

and therefore when the secant becomes a tangent,

$$PO = PO',$$

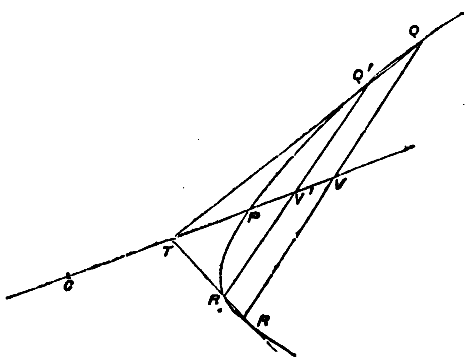
$$\therefore\ RQ \,.\, Qr = PO^2.$$

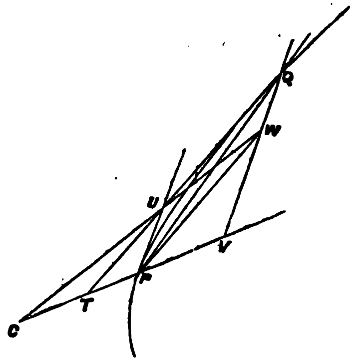
COR. 1. *The diameter* CP *bisects all the chords parallel to the tangent at* P.

Since $PO = PO'$, $RV = rV$,

and $\therefore\ QV = qV$.

COR. 2. Hence $CV \,.\, CT = CP^2$ as in the ellipse.





The proof is the same as that given for the ellipse in Th. 11, Cor. 2.

THEOREM 17. ORDINATE AND ABSCISSA PARALLEL TO ASYMPTOTE.

The rectangle contained by the ordinate and abscissa of any point, measured from the centre parallel to the asymptotes,

$$=\frac{1}{4}(AC^2+BC^2).$$

Draw WPW' perpendicular to the axis.

Then, by similar triangles,

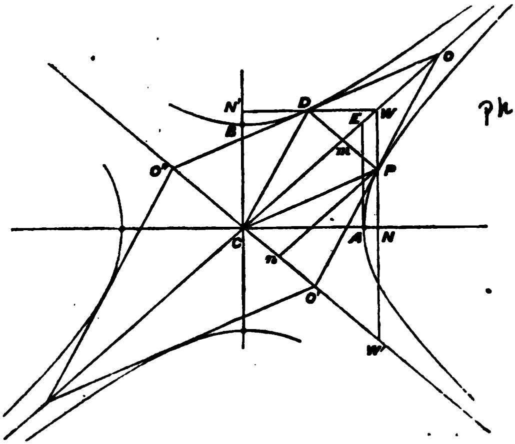
$$Pn : PW' :: CE : 2AE, \quad \text{∴ CW : WW'}$$

and
$$Pm : PW :: CE : 2AE,$$

$$\therefore Pn \,.\, Pm : PW \,.\, PW' :: CE^2 : 4AE^2,$$

$$\therefore Pn \,.\, Pm : BC^2 :: AC^2+BC^2 : 4BC^2, \quad \text{Th. 15.}$$

$$\therefore Pn \,.\, Pm = \frac{1}{4}(AC^2+BC^2).$$



Pk : Pn' :: Wc : Wn

Cor. *Hence the parallelogram* PC *and the triangle* OCO′ *are of constant area.*

THEOREM 18. CONJUGATE HYPERBOLA.

Tangents at the extremities of conjugate diameters intersect on the asymptotes, and form a parallelogram of constant area = 4AC . BC.

Let OPO', ODO'' be tangents from O a point on the asymptote, meeting the hyperbola and its conjugate in P, D, and making therefore $OP = PO'$ and $OD = DO''$, and therefore PmD parallel to $O'O''$.

Then by the last theorem

$$Pm \,.\, mC = \frac{1}{4}(AC^2 + BC^2),$$

$$Dm \,.\, mC = \frac{1}{4}(BC^2 + AC^2)\,;$$

$$\therefore Pm = Dm,$$

and because $$OP = PO',$$

$$\therefore \text{ also } Om = mC;$$

and therefore $QDCP$ is a parallelogram;

$$\therefore CD, CP \text{ are conjugate diameters.}$$

Moreover the area $OO'O''$, which is half the parallelogram formed by the four tangents at the extremities of conjugate diameters, is constant, $= 4PmCn$.

But when the tangents are at the vertices the parallelogram becomes a rectangle $= 2AC \times 2BC$;

$$\therefore \text{ the parallelogram} = 4AC \,.\, BC.$$

COR. 1. *If* PK *is perpendicular to* CD,

$$PK \,.\, CD = AC \,.\, BC.$$

COR. 2. *Since* PD *is bisected in* m, *and the asymptotes are equally inclined to the axes, therefore parallels to the axes through* P *and* D *intersect on the asymptote.*

For the intercept mW made by both the parallels is equal to mP or mD.

COR. 3. *Hence also* DN′ : PN :: AC : BC,

and CN′ : CN :: BC : AC.

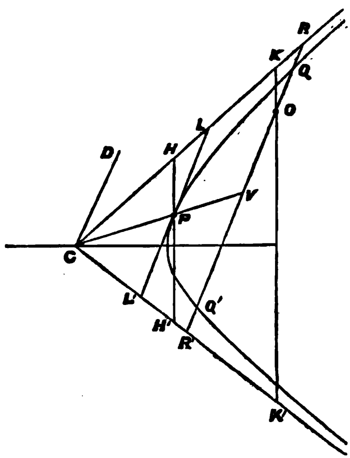
COR. 4. $CP^2 - CD^2 = AC^2 - BC^2$.

For determining CP^2 and CD^2 from the triangles CPm, CDm, their difference is proportional to $Cm \,.\, mP$, or is constant, and therefore $= AC^2 - BC^2$.

THEOREM 19. OBLIQUE ORDINATES AND ABSCISSÆ.

In the hyperbola $QV^2 : PV \cdot P'V :: CD^2 : CP^2$.

Let QV meet the asymptotes in R, R'.



Then $QV^2 = RV^2 - RQ \cdot QR' = RV^2 - CD^2$, (Th. 15.)

because $PL = CD$. (Th. 18.)

But $RV : CV : PL : CP$;

$$\therefore RV^2 : CD^2 :: CV^2 : CP^2;$$

$$\therefore RV^2 - CD^2 : CD^2 :: CV^2 - CP^2 : CP^2,$$

or $QV^2 : CD^2 :: PV \cdot P'V : CP^2$,

and $\therefore QV^2 : PV \cdot P'V :: CD^2 : CP^2$.

THEOREM 20. RECTANGLES CONTAINED BY SEGMENTS OF INTERSECTING CHORDS.

If two chords of a hyperbola intersect one another, the rectangles contained by their segments are proportional to the squares of the diameters parallel to them.

Let QOQ' be one of the chords through O, in the figure of Theorem 19, meeting the asymptotes in R, R', CD the parallel diameter,

then will $QO \,.\, OQ'$ be proportional to CD^2.

Draw CPV the diameter to bisect QQ'.

Since $$QO \,.\, OQ' = QV^2 - OV^2,$$
and $$RO \,.\, OR' = RV^2 - OV^2;$$
$$\therefore\ RO \,.\, OR' - QO \,.\, OQ' = RV^2 - QV^2$$
$$= RQ \,.\, QR'$$
$$= CD^2;$$
$$\therefore\ QO \,.\, OQ' = RO \,.\, OR' - CD^2.$$

But if KOK' be drawn through O, and HPH' through P, perpendicular to the transverse axis, meeting the asymptotes in K, K', H, H', since
$$RO : KO :: LP : HP,$$
and $$R'O : K'O :: L'P : H'P;$$
$$\therefore\ RO \,.\, OR' : KO \,.\, OK' :: CD^2 : BC^2;$$
therefore while O remains fixed, and therefore $KO \,.\, OK'$ does not alter, $RO \,.\, OR'$ is proportional to CD^2,

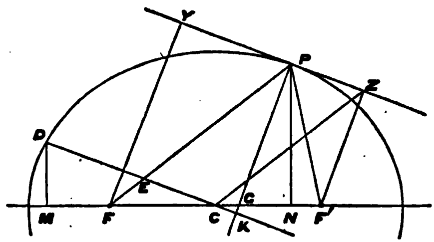
and therefore also $QO \,.\, OQ'$ is proportional to CD^2.

Obs. If one or both of the chords becomes a tangent, the rectangle contained by the segments of the chords becomes the square of the tangent.

THEOREM 21. PRODUCT OF FOCAL DISTANCES.

In any central conic

$$FP \,.\, F'P = CD^2.$$



Let FP cut CD in E; then $PE = CZ = AC$; and since by similar triangles

$$FP : FY :: PE : PK,$$

and $$F'P : F'Z :: PE : PK;$$

$$\therefore FP \,.\, F'P : FY \,.\, F'Z :: PE^2 : PK^2,$$

but $FY \,.\, F'Z = BC^2$, and $PE = AC$;

$$\therefore FP \,.\, F'P : BC^2 :: AC^2 : PK^2;$$

but $PK \,.\, CD = AC \,.\, BC$, (Th. 13 and 18.)

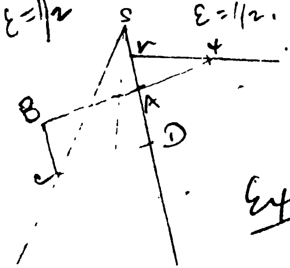
$$CD^2 : BC^2 :: AC^2 : PK^2;$$

$$\therefore FP \,.\, F'P : BC^2 :: CD^2 : BC^2;$$

$$\therefore FP \,.\, F'P = CD^2.$$

V & A are any pts. Make AY = 2 AV, then ...

$\mathcal{E} = 1/2$



$\mathcal{E} = 1/2$. Draw AB = [illegible] major axis
" BC ∥ to VA
" CD " " BA.
CD is the major axis
[illegible]

Ex 2.

EXERCISES.

To draw a tang from pt outside. See the S. P125.

1. In the figure of Theorem 1, prove that

$$FF' = AB' \text{ and } AA' = VU.$$

2. Shew how to cut from a given cone an ellipse of given axis and eccentricity.

3. Give some mechanical contrivance for describing an ellipse and hyperbola.

4. Prove that

$$CF : CX :: FC^2 : AC^2$$

in any central conic.

5. Prove that in the ellipse $FP + F'P$ is greater or less than AA', according as P is outside or inside the ellipse. What is the corresponding property of the hyperbola?

6. If a circle be described on the axis minor of an ellipse as diameter, and PQM, parallel to the axis major, meet the ellipse in P, the circle in Q and axis minor in M, prove that

$$QM : PM :: BC : AC.$$

7. A circle is described to touch two unequal intersecting circles, prove that the locus of its centre consists of a confocal ellipse and hyperbola.

8. If a hyperbola and ellipse are confocal, they cut one another at right angles.

9. On AB is described a segment of a circle, which is trisected in P, Q. Find the locus of P.

10. Prove that the two tangents drawn to a central conic from any point are in the ratio of the parallel diameters.

11. Prove that the locus of the point from which tangents can be drawn at right angles to a central conic is a circle whose radius is

$$\sqrt{AC^2 \pm BC^2},$$

the upper sign being taken for the ellipse, and the lower for the hyperbola.

12. Prove that the tangent at the extremity of the latus rectum intersects the axis major at the foot of the directrix, and the axis minor at a point T, such that

$$CT = CA.$$

13. Prove that

$$CP + CD > AC + BC,$$

and $$CP \sim CD < AC - BC.$$

14. Given a central conic to find its centre and axes, foci and directrix.

If an arc of a conic section is given, shew how to find its species.

15. A quadrilateral figure circumscribes an ellipse, prove that its pairs of opposite sides subtend angles at either focus whose sum is two right angles.

16. A circle touches an ellipse in P, and cuts it in Q, R, prove that PQ, PR are equally inclined to the axis.

17. If T is the point of intersection of the tangent at P with the tangent at A, prove that FT bisects the angle AFP. Hence find the locus of the centres of the escribed circles of the triangle FPF'.

18. If NP produced meet the tangent at the extremity of the latus rectum in T, $TN = FP$.

19. Ellipses are described with a given focus, and to touch a given straight line in a given point, find the locus of the other focus and of the centre.

20. Ellipses are described with a given focus, and axis major of given length, to touch a given straight line: find the locus of the other focus, and centre. Ans. A circle.

21. Ellipses are described with a given focus, and axis minor of given length, to touch a given straight line: to find the locus of the other focus.

Ans. A straight line parallel to the given straight line.

22. If from the extremities of the axes of an ellipse any four parallel straight lines be drawn, they will intersect the ellipse in the extremities of conjugate diameters.

23. Prove that in an ellipse AP, $A'P$ are parallel to a pair of conjugate diameters, P being any point on the curve.

24. A line PFG is constrained to move so that two fixed points in it, F and G, lie on two axes at right angles to one another. Shew that the locus of P is an ellipse.

Hence obtain a mechanical means of drawing an ellipse with given axes.

25. An ellipse slides between two lines at right angles to one another; find the locus of its centre. (Th. 5, Cor.)

26. The locus of the points of bisection of chords of an ellipse drawn through a given point is an ellipse of equal eccentricity.

27. If the focus of a conic and two points on the curve be given, prove that its directrix will pass through a fixed point. (Th. 3.)

28. Given three tangents to an ellipse and one focus, find the other focus. (Th. 4. Cor. 1.)

29. Prove that the circle FPF' passes through the points of intersection of the tangent and normal at P with the minor axis.

30. If CE parallel to the tangent at P meets FP in E, and gE is joined, gE is perpendicular to FP. (Th. 8.)

31. With a given focus, and three given points on the curve, find the other focus.

32. The locus of the foot of the perpendicular from the centre on any chord that subtends a right angle at the centre is a circle.

33. Shew that the areas of the ellipse and its auxiliary circle are to one another as $CB : CA$.

34. Chords are drawn to a conic from a fixed point; shew that tangents at their extremities intersect on a fixed straight line.

35. A rifle bullet hits a target. Find the locus of places at which the sound of the discharge and of the hit are heard at the same instant.

36. Given the asymptotes, and one point on the curve, construct for the foci.

37. The corner of a leaf is turned down so that the triangle is of constant area. Find the locus of its middle point.

38. Prove by the method of projections that ellipses of equal eccentricity, and whose axes are parallel, can intersect in only two points.

39. A straight line is drawn through a fixed point and is terminated by two given straight lines: find the locus of its middle point.

40. If the directrix and focus of an ellipse be fixed, and its axis major continually increased, prove that in the limit the ellipse becomes a parabola. Hence obtain the tangent property of the parabola.

41. The locus of the intersection of tangents to an ellipse at right angles to one another is a circle. Deduce the corresponding property in the parabola.

42. The semi-latus rectum is a harmonic mean between the segments of any focal chord.

43. If e, e' are the eccentricities of a hyperbola and its conjugate, prove that

$$e \cdot AC = e' \cdot BC.$$

44. If F, f are the foci of a hyperbola, and its conjugate, and P, P' conjugate points on the hyperbola and its conjugate,

$$fP' - FP = AC - BC, \text{ and } CF = Cf.$$

45. If any two tangents be drawn to a hyperbola, the lines that join the points where they cut the asymptotes will be parallel.

46. If an ellipse is described with a fixed centre to touch two given straight lines, the locus of its focus is a hyperbola.

47. FY is drawn to make a constant angle FYP with the tangent at P; shew that the locus of Y is a circle.

48. If GK is the perpendicular on SP from G, the foot of the normal at P, PK will be equal to half the latus rectum.

49. A chord of an ellipse which subtends a constant angle at the focus always touches an ellipse with the same focus and directrix.

50. A circle is inscribed in a triangle; prove that if an ellipse be described to touch the three sides of the triangle, and one of its foci is on this circle, the other will be on the same circle.

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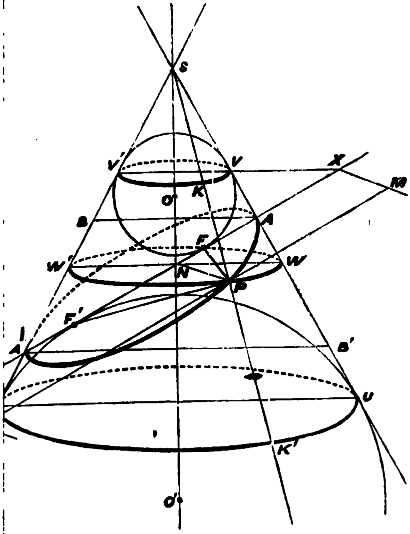
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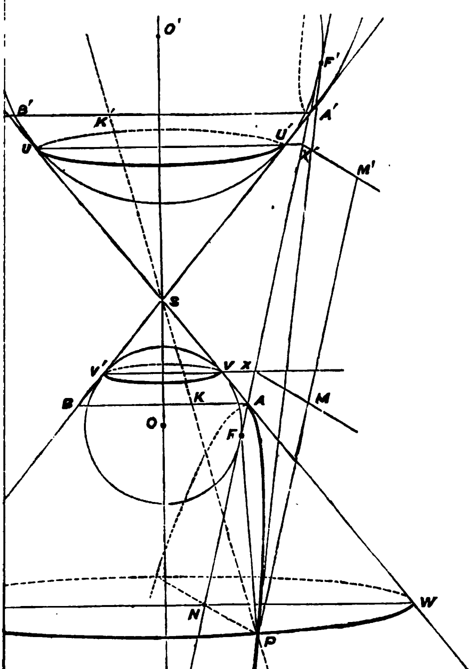
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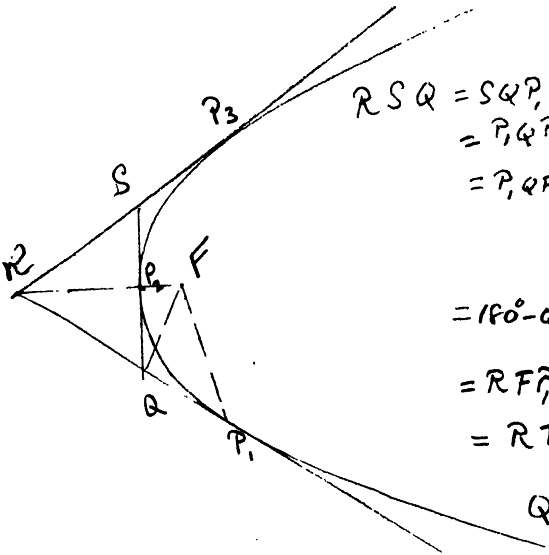

$$
\begin{aligned}
RSQ &= SQP_1 - SRQ \\
&= P_1QP_2 - P_1RP_3 \\
&= P_1QF + FQP_2 - (P_1RF + FRP_3) \\
&\qquad \underset{FP_1Q}{\text{''Th 7}} \qquad \underset{RP_1F}{\text{Th 7 ''}} \\
&= 180^\circ - QFP_1 - (180^\circ - RFP_1) \\
&= RFP_1 - QFP_1 \\
&= RFQ.
\end{aligned}
$$

Q.E.D.